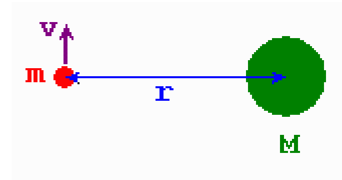
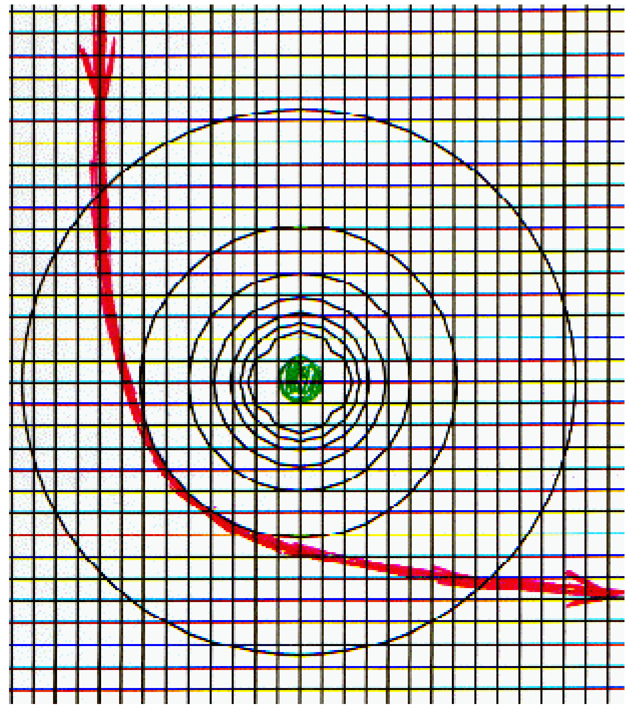
DAY 7

Summary of Topics Covered in Today's Lecture

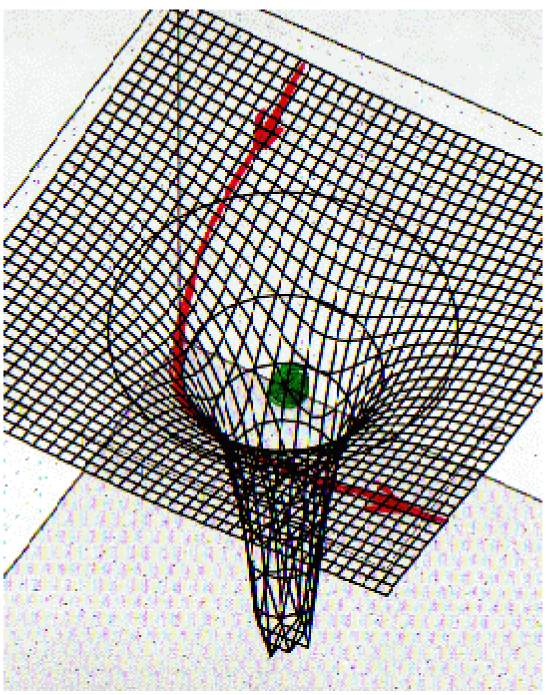


Hyperbolic Trajectories

If the system has even more energy that that needed for *m* to have a parabolic trajectory, then *m* will have a hyperbolic trajectory. *m* does not even remotely begin to orbit *M* – rather, *M*'s "gravitational well" only deflects *m* as *m* passes by at high speed.



The figures show a Hyperbolic Trajectory (in red) around central mass (in green) plotted on an equipotential plot and a surface plot of the potential function of the central mass showing the hyperbolic trajectory.



The total energy in the system is

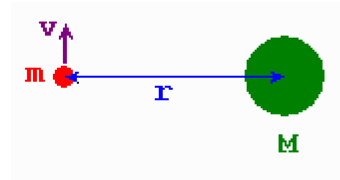
$$E_{hyperbolic} = KE + PE > 0$$

Thus the energy of an object in a hyperbolic trajectory is said to be positive, or greater than zero. A hyperbolic trajectory will take an object to infinity and then some.

Hyperbolic trajectory --
"To Infinity… and Beyond!"

Summary

So, if a small object *m* is located a distance r from a much larger object *M* which does not move, and we launch *m* with speed *v* as shown in the figure at left, then the path *m* follows will depend on the speed *v*

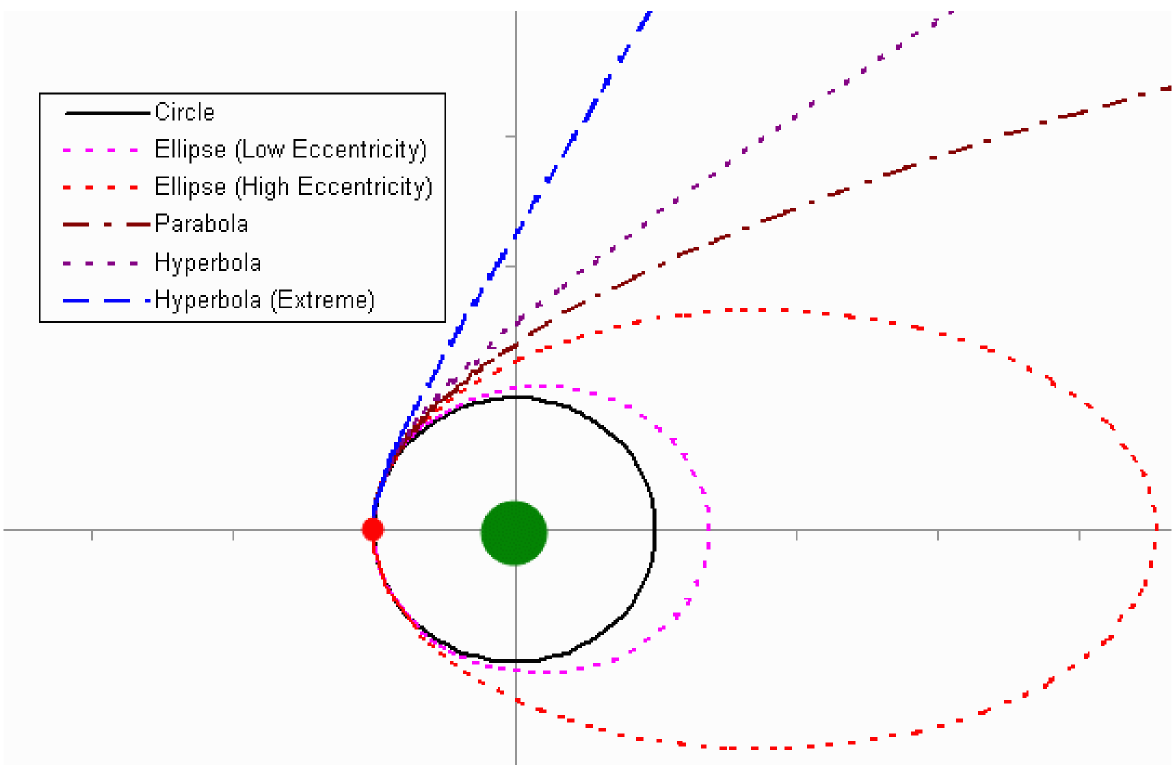


- ☑ If $v = (GM/r)^{1/2}$ then *m* will orbit *M* in a circular path as shown by the black orbit in the figure below. The energy of *m* is $E_{circular} = -\frac{1}{2}GMm/r < 0$

- ☑ If $(GM/r)^{1/2} < v < (2GM/r)^{1/2}$ then *m* will orbit *M* in an elliptical path. The higher *v* is (within this range), the more eccentric the shape of the ellipse will be (see figure below) and the further out *m* will move from *M* at its most extreme point. The energy of *m* is $-\frac{1}{2}GMm/r < E_{elliptical} < 0$

- ☑ If $v = (2GM/r)^{1/2}$ then *m* will travel away from *M* on a parabolic trajectory as shown by the dashed brown trajectory in the figure below. The energy of *m* is $E_{parabolic} = 0$.

- ☑ If $v > (2GM/r)^{1/2}$ then *m* will travel away from *M* on a hyperbolic trajectory. The more extreme *v* is, the flatter the trajectory will be. The energy of *m* is $E_{hyperbolic} > 0$.



A Gravity Example That Really Pushes Our Limits, Mathematically

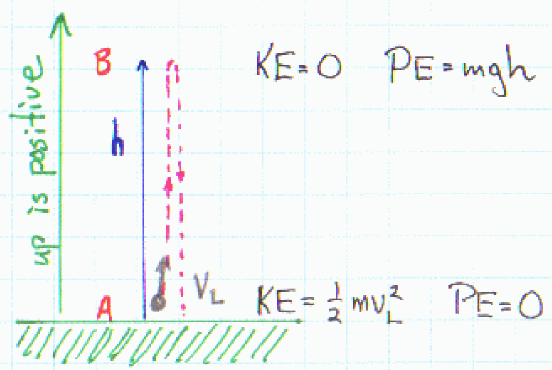
Today we did a series of example problems that started off being pretty easy but ended up being quite difficult. Note, however, that they all use the same concepts.

Example Problem #1

An object of mass m is launched upward with speed v_L from the Earth's surface. The object stays close to Earth where the gravitational field strength **g** is a constant. Derive an equation for the height the object reaches. Derive an equation for the time the object is in the air. Assume air friction does not convert a significant amount of the object's energy to heat.

Solution:

In this problem I use the old PE = mgy formula for gravitational potential energy that is based on the assumption that the Earth's gravitational field is constant.



Energy at A and Energy at B must be the same:

$$E_A = E_B$$

$$KE_A + PE_A = KE_B + PE_B$$

$$\tfrac{1}{2}mv_L^2 + 0 = 0 + mgh$$

$$\tfrac{1}{2}\cancel{m}v_L^2 = \cancel{m}gh$$

$$h = \frac{v_L^2}{2g}$$

Height reached by object.

To find time, use kinematic equation

$$V = V_0 + at$$

$V_0 = V_L$ at A
$V = 0$ at B
$a = -g$ $(-9.8\,m/s^2)$

This will find time to go from A→B.

$$0 = V_L - gt$$

$$t = \frac{V_L}{g}$$

A→B
The time from A→B→A (up & back down) is twice this:

$$t_{TOT} = 2\frac{V_L}{g}$$

Example Problem #2

An object is launched upward from the Earth's surface at 30 m/s. How high will it go and how long will it be in the air?

Solution:

$$h = \frac{v_L^2}{2g} = \frac{\left(30\frac{m}{s}\right)^2}{2(9.8m/s^2)} = 45.9m$$

$$t_{TOT} = \frac{2v_L}{g} = \frac{2(30m/s)}{9.8m/s^2} = 6.12s$$

Example Problem #3

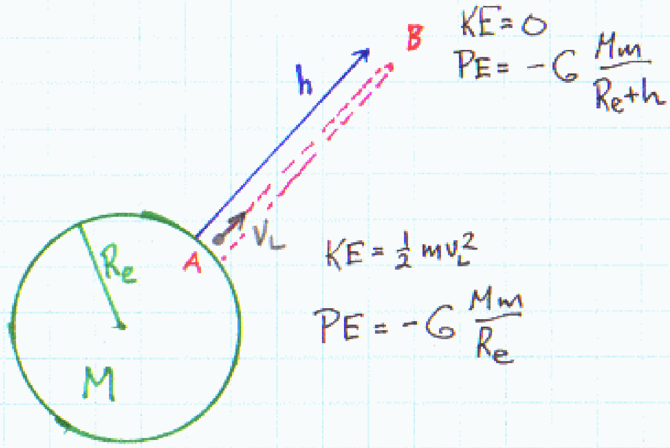
An object of mass *m* is launched upward with speed v_L from the Earth's surface (or from just above the Earth's atmosphere). Derive an equation for the height the object reaches – no matter how high the object goes. Derive an equation for the time the object is in the air. Assume air friction does not convert a significant amount of the object's energy to heat.

Solution:

Now I can't assume the gravitational field is constant any more. I have to use the new PE formula from Day 6 for a point mass

$$PE = -G\frac{Mm}{r}$$

However, the basic idea is still the same.



Energy at A and B must be the same

$$E_A = E_B$$

$$KE_A + PE_A = KE_B + PE_B$$

$$\frac{1}{2}mv_L^2 - \frac{GMm}{R_e} = 0 - \frac{GMm}{R_e+h}$$

$$\frac{1}{2}\cancel{m}v_L^2 = \frac{GM\cancel{m}}{R_e} - \frac{GM\cancel{m}}{R_e+h}$$

$$V_L^2 = 2GM\left(\frac{1}{R_e} - \frac{1}{R_e+h}\right)$$

$$\frac{V_L^2}{2GM} = \frac{1}{R_e} - \frac{1}{R_e+h}$$

$$\frac{1}{R_e+h} = \frac{1}{R_e} - \frac{V_L^2}{2GM}$$

$$R_e+h = \left(\frac{1}{R_e} - \frac{V_L^2}{2GM}\right)^{-1}$$

$$h = \left(\frac{1}{R_e} - \frac{V_L^2}{2GM}\right)^{-1} - R_e$$

Height Reached by object.

Example Problem #4

An object is launched upward from the Earth's surface at 30 m/s. How high will it go, using this formula?

Solution:

The mass of Earth is $M = 5.98\times10^{24}$ kg

The radius of Earth is $r_E = 6.37\times10^{6}$ m

$G = 6.67\times10^{-11}\ \mathrm{Nm^2/kg^2}$

$V_L = 30$ m/s

$$h = \left(\frac{1}{R_e} - \frac{V_L^2}{2GM}\right)^{-1} - R_e$$

$$h = \left(\frac{1}{6.37\times10^{6}} - \frac{30^2}{2(6.67\times10^{-11})(5.98\times10^{24})}\right)^{-1} - 6.37\times10^{6}$$

$$h = \left(1.5698587127\times10^{-7} - 1.1281984426\times10^{-12}\right)^{-1} - 6.37\times10^{6}$$

$$h = 6370045.7791243823 - 6.37\times10^{6} = 45.7791243823$$

Working the units out I get

$$h = \left(\frac{1}{\mathrm{m}} - \frac{\frac{\mathrm{m^2}}{\mathrm{s^2}}}{\mathrm{N}\frac{\mathrm{m^2}}{\mathrm{kg^2}}\mathrm{kg}}\right)^{-1} - \mathrm{m} = \left(\frac{1}{\mathrm{m}} - \frac{\frac{1}{\mathrm{s^2}}}{\mathrm{N}\frac{1}{\mathrm{kg}}}\right)^{-1} - \mathrm{m} = \left(\frac{1}{\mathrm{m}} - \frac{\frac{1}{\mathrm{s^2}}}{\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s^2}}\frac{1}{\mathrm{kg}}}\right)^{-1} - \mathrm{m} = \left(\frac{1}{\mathrm{m}} - \frac{1}{\mathrm{m}}\right)^{-1} - \mathrm{m} = \mathrm{m} - \mathrm{m} = \mathrm{m}$$

So there I have it – $h = 45.8$ m. Basically the same result as before.

Example Problem #5

An object is launched upward from the Earth's surface at 10,000 m/s. How high will it go, using this formula?

Solution:

The mass of Earth is $M = 5.98\times10^{24}$ kg
The radius of Earth is $r_E = 6.37\times10^{6}$ m
$G = 6.67\times10^{-11}$ Nm^2/kg^2
$V_L = 10{,}000$ m/s

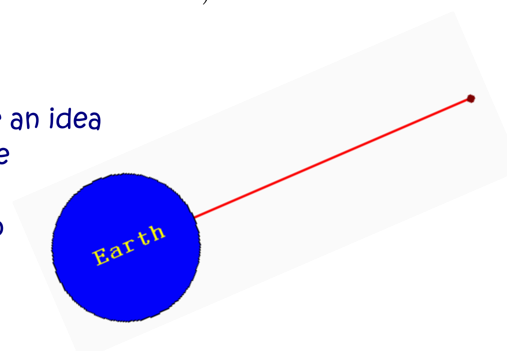
$$h = \left(\frac{1}{R_e} - \frac{V_L^2}{2GM}\right)^{-1} - R_e$$

$$h = \left(\frac{1}{6.37\times10^{6}} - \frac{10000^2}{2(6.67\times10^{-11})(5.98\times10^{24})}\right)^{-1} - 6.37\times10^{6}$$

$$h = \left(1.5698587127158555729984301412873\times10^{-7} - 1.2535538250941418922645700561091\times10^{-7}\right)^{-1} - 6.37\times10^{6}$$

$$h = 31615066.32 - 6.37\times10^{6} = 25245066\ m$$

So the object will reach a height of 25.2 million meters. To give an idea how far that it, the radius of Earth is 6.37 million meters, so the diameter is 12.74 million meters. Therefore if the object is launched from near Earth's surface at that speed, it will rise to about 2 Earth diameters above the surface, as shown in the figure at right.



Example Problem #6

What speed is needed to launch an object out to the distance of the Moon's orbit, and how long will it take to go out and come back?

Solution:

The mass of Earth is $M = 5.98\times10^{24}$ kg
The radius of Earth is $r_E = 6.37\times10^{6}$ m
$G = 6.67\times10^{-11}$ Nm^2/kg^2

Distance of Moon's orbit = 3.84×10^{8} m so $R_e+h = 3.84 \times 10^{8}$ m

$$h = \left(\frac{1}{R_e} - \frac{V_L^2}{2GM}\right)^{-1} - R_e$$

First I re-write this in terms of V_L:

$$V_L = \sqrt{2GM\left(\frac{1}{R_e} - \frac{1}{R_e + h}\right)}$$

and now let's "plug-n-chug":

$$V_L = \sqrt{2(6.67\times10^{-11})(5.98\times10^{24})\left(\frac{1}{6.37\times10^{6}} - \frac{1}{3.84\times10^{8}}\right)} = 11097.5\,m/s$$

So it must be launched at a speed of 11,100 m/s (24,800 mph) to reach the distance of the moon's orbit.

Figuring the time is a tougher issue. I have to use a spreadsheet to estimate the time. None of these answers depend on mass, but I'll need to pick a mass for my spreadsheet. Let's say 100 kg. Now I'll figure the total energy the object has at Earth's surface when it is launched at 11,097.5 m/s.

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B2

TOTAL ENERGY (CALCULATED AT LAUNCH)		
m =	100	kg
vL =	11097.5	m/s
KE = 1/2 mv^2 =	6.1577E+09	J
M =	5.98E+24	kg
Re =	6.37E+06	m
G =	6.67E-11	Nm^2/kg^2
PE = - G M m/Re =	-6.2616E+09	J
E = KE + PE =	-1.0391E+08	J

Note that the total energy is negative, as it should be if the object is not going to escape Earth totally.

Now, for a variety of heights I calculate the PE of the object. Note that the first row, with h=0, gives me my original PE. I'm incrementing my heights in steps of 10,000,000 m.

E19

E = KE + PE = -1.0391E+08 J

h (m)	r (m)	PE (J)
0	6.37E+06	-6.2616E+09
10,000,000	1.64E+07	-2.4366E+09
20,000,000	2.64E+07	-1.5126E+09
30,000,000	3.64E+07	-1.0967E+09
40,000,000	4.64E+07	-8.6018E+08
50,000,000	5.64E+07	-7.0759E+08
60,000,000	6.64E+07	-6.0097E+08
70,000,000	7.64E+07	-5.2228E+08
80,000,000	8.64E+07	-4.6181E+08
90,000,000	9.64E+07	-4.1389E+08
100,000,000	1.06E+08	-3.7498E+08
110,000,000	1.16E+08	-3.4276E+08
120,000,000	1.26E+08	-3.1563E+08
130,000,000	1.36E+08	-2.9249E+08
140,000,000	1.46E+08	-2.7251E+08
150,000,000	1.56E+08	-2.5508E+08
160,000,000	1.66E+08	-2.3975E+08

Now I add a column for the total energy (which does not change), and take the difference between the total energy and the PE to get the KE.

E = KE + PE = -1.0391E+08 J

h (m)	r (m)	PE (J)	E (J)	KE (J)
0	6.37E+06	-6.2616E+09	-1.0391E+08	6.1577E+09
10,000,000	1.64E+07	-2.4366E+09	-1.0391E+08	2.3327E+09
20,000,000	2.64E+07	-1.5126E+09	-1.0391E+08	1.4087E+09
30,000,000	3.64E+07	-1.0967E+09	-1.0391E+08	9.9278E+08
40,000,000	4.64E+07	-8.6018E+08	-1.0391E+08	7.5627E+08
50,000,000	5.64E+07	-7.0759E+08	-1.0391E+08	6.0368E+08
60,000,000	6.64E+07	-6.0097E+08	-1.0391E+08	4.9707E+08
70,000,000	7.64E+07	-5.2228E+08	-1.0391E+08	4.1837E+08
80,000,000	8.64E+07	-4.6181E+08	-1.0391E+08	3.5790E+08
90,000,000	9.64E+07	-4.1389E+08	-1.0391E+08	3.0998E+08
100,000,000	1.06E+08	-3.7498E+08	-1.0391E+08	2.7107E+08
110,000,000	1.16E+08	-3.4276E+08	-1.0391E+08	2.3885E+08
120,000,000	1.26E+08	-3.1563E+08	-1.0391E+08	2.1173E+08
130,000,000	1.36E+08	-2.9249E+08	-1.0391E+08	1.8858E+08
140,000,000	1.46E+08	-2.7251E+08	-1.0391E+08	1.6860E+08
150,000,000	1.56E+08	-2.5508E+08	-1.0391E+08	1.5117E+08
160,000,000	1.66E+08	-2.3975E+08	-1.0391E+08	1.3584E+08
170,000,000	1.76E+08	-2.2615E+08	-1.0391E+08	1.2225E+08

Since $KE = \frac{1}{2}mv^2$
I can solve for v … $(v = 2KE/m)^{\frac{1}{2}}$
I add a column for that, too.

PE = - G M m/Re = -6.2616E+09 J

E = KE + PE = -1.0391E+08 J

h (m)	r (m)	PE (J)	E (J)	KE (J)	v (m/s)
0	6.37E+06	-6.2616E+09	-1.0391E+08	6.1577E+09	11097.5
10,000,000	1.64E+07	-2.4366E+09	-1.0391E+08	2.3327E+09	6830.314
20,000,000	2.64E+07	-1.5126E+09	-1.0391E+08	1.4087E+09	5307.857
30,000,000	3.64E+07	-1.0967E+09	-1.0391E+08	9.9278E+08	4455.967
40,000,000	4.64E+07	-8.6018E+08	-1.0391E+08	7.5627E+08	3889.149
50,000,000	5.64E+07	-7.0759E+08	-1.0391E+08	6.0368E+08	3474.704
60,000,000	6.64E+07	-6.0097E+08	-1.0391E+08	4.9707E+08	3152.986
70,000,000	7.64E+07	-5.2228E+08	-1.0391E+08	4.1837E+08	2892.658

Now I look at this. This is just an estimate, but for the 1st 10,000,000 m of travel, the object starts at 11,097.5 m/s and ends at 6830.314122 m/s. The average speed here is 8963.907061 m/s. With a distance of 10,000,000 m and an average speed of 8963.907061 m/s, I can figure the time to cover that first 10,000,000 m is

t = 10,000,000 m /(8963.907061 m/s) = 1115.585 s

I'll so this for every 10,000,000 m increment:

h (m)	r (m)	PE (J)	E (J)	KE (J)	v (m/s)	t (s)
0	6.37E+06	-6.2616E+09	-1.0391E+08	6.1577E+09	11097.5	
10,000,000	1.64E+07	-2.4366E+09	-1.0391E+08	2.3327E+09	6830.314	1115.585
20,000,000	2.64E+07	-1.5126E+09	-1.0391E+08	1.4087E+09	5307.857	1647.695
30,000,000	3.64E+07	-1.0967E+09	-1.0391E+08	9.9278E+08	4455.967	2048.378
40,000,000	4.64E+07	-8.6018E+08	-1.0391E+08	7.5627E+08	3889.149	2396.611
50,000,000	5.64E+07	-7.0759E+08	-1.0391E+08	6.0368E+08	3474.704	2715.97
60,000,000	6.64E+07	-6.0097E+08	-1.0391E+08	4.9707E+08	3152.986	3017.643
70,000,000	7.64E+07	-5.2228E+08	-1.0391E+08	4.1837E+08	2892.658	3308.167
80,000,000	8.64E+07	-4.6181E+08	-1.0391E+08	3.5790E+08	2675.457	3591.88
90,000,000	9.64E+07	-4.1389E+08	-1.0391E+08	3.0998E+08	2489.911	3871.941

I extend the table on down until I reach the height of the orbit of the moon (where r = 3.84×10^8 m). Going beyond that is asking the spreadsheet to take the square root of a negative number, which it will not do.

340,000,000	3.46E+08	-1.1516E+08	-1.0391E+08	1.1249E+07	474.3144	19684.67
350,000,000	3.56E+08	-1.1192E+08	-1.0391E+08	8.0173E+06	400.4334	22863.73
360,000,000	3.66E+08	-1.0887E+08	-1.0391E+08	4.9624E+06	315.0359	27953.68
370,000,000	3.76E+08	-1.0598E+08	-1.0391E+08	2.0698E+06	203.4579	38573.26
380,000,000	3.86E+08	-1.0323E+08	-1.0391E+08	##########	#NUM!	#NUM!

Now I have a huge list of times – one for every 10,000,000 m of travel. I add up all these to get the time going up to the Moon's orbit.

=SUM(H19:H55)

C	D	E	F	G	H	I	J
M m/Re =	-6.2616E+09 J						
PE =	-1.0391E+08 J						
r (m)	PE (J)	E (J)	KE (J)	v (m/s)	t (s)		SUM of ALL t values
6.37E+06	-6.2616E+09	-1.0391E+08	6.1577E+09	11097.5			19:H55)
1.64E+07	-2.4366E+09	-1.0391E+08	2.3327E+09	6830.314	1115.585		
2.64E+07	-1.5126E+09	-1.0391E+08	1.4087E+09	5307.857	1647.695		

My answer for the time going up is 343283.0786 sec. The total time is then 686566.1572 sec or 8 days.

This is only an estimate. If I make my increments smaller the estimate gets better. Using 1,000,000 m increments I get about 9.5 days, which is close to the true value.

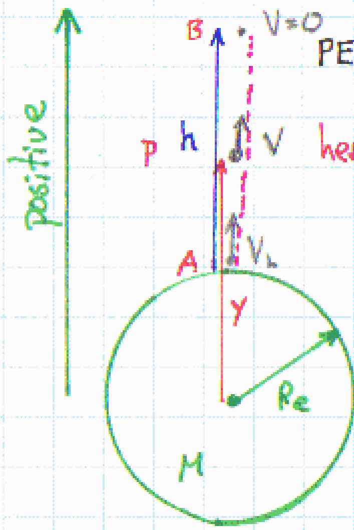
FOR PHYSICS 232 ONLY

To get a true value requires the use of calculus. I'll use the basic definition of v and combine it with the energy stuff I'll been doing so far:

$$v = \frac{dy}{dt}$$

$$dy = v\,dt$$

If I can get v as a function of y I can do an integral.



$PE = -\frac{GMm}{R_e+h}$ $KE = 0$

here $g = -G\frac{M}{y^2}$ $KE = \frac{1}{2}mv^2$ $PE = -\frac{GMm}{y}$

$$E_p = E_B$$

$$KE_p + PE_p = KE_B + PE_B$$

$$\frac{1}{2}mv^2 - \frac{GMm}{y} = 0 - \frac{GMm}{R_e+h}$$

Like before $\Rightarrow$ $v^2 = 2GM\left(\frac{1}{y} - \frac{1}{R_e+h}\right)$

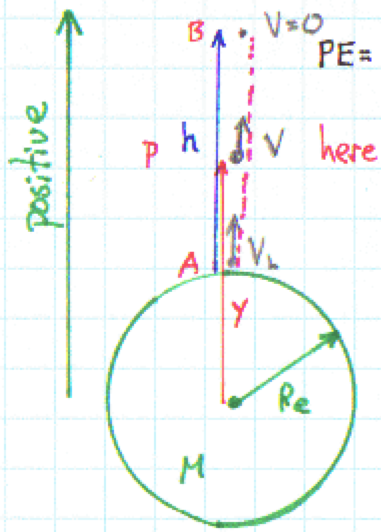
$$v = \sqrt{2GM}\sqrt{\frac{1}{y} - \frac{1}{R_e+h}}$$

Now $dy = \sqrt{2GM}\sqrt{\frac{1}{y} - \frac{1}{R_e+h}}\;dt$

How to find time now? Well, g varies so I can't use my constant g kinematic equations. Let's go back to basics

$$V = \frac{dy}{dt}$$

$$dy = V\,dt$$ If I can get v as a function of y I can do an integral.



$$PE = -\frac{GMm}{R_e+h} \qquad KE = 0$$

here $g = -G\frac{M}{y^2}$ $\quad KE = \frac{1}{2}mv^2 \quad PE = -\frac{GMm}{y}$

$$E_p = E_B$$

$$KE_p + PE_p = KE_B + PE_B$$

$$\frac{1}{2}mv^2 - \frac{GMm}{y} = 0 - \frac{GMm}{R_e+h}$$

Like before → $$V^2 = 2GM\left(\frac{1}{y} - \frac{1}{R_e+h}\right)$$

$$V = \sqrt{2GM}\sqrt{\frac{1}{y} - \frac{1}{R_e+h}}$$

Now $$dy = \sqrt{2GM}\sqrt{\frac{1}{y} - \frac{1}{R_e+h}}\;dt$$

$$\frac{dy}{\sqrt{2GM}\sqrt{\frac{1}{y}-\frac{1}{R_e+h}}} = dt$$

Now let's integrate this to find t!

$$\frac{1}{\sqrt{2GM}}\int \frac{dy}{\sqrt{\frac{1}{y}-\frac{1}{R_e+h}}} = \int dt$$

Limits of integration:

At launch (A) $t=0$ $y=R_e$

At highest point (B) $t=t$ $y=R_e+h$

$$\frac{1}{\sqrt{2GM}}\int_{R_e}^{R_e+h} \frac{dy}{\sqrt{\frac{1}{y}-\frac{1}{R_e+h}}} = \int_0^t dt = t$$

I'm going to feed this integral to some computer integration program.

$R_e+h = 3.84 \times 10^8$ m = 384,000,000 m
$R_e = 6.37 \times 10^6$ m = 6,370,000 m

Plugging the numbers in I get the integral to be 1.18×10^{13}. I still include the value in front of the integral so

$$\frac{1.1809\times10^{13}}{\sqrt{2GM}} = 4.18\times10^{5}\ s$$

For the full trip (up and down), we get $8.36\text{x}10^5$ seconds or 9.7 days! Years ago I did this problem with a class and we had to use a TI-92 calculator to do the integration and it was much tougher, and before that this sort of problem was almost impossible. Isn't technology wonderful?