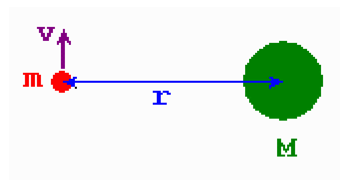
DAY 6

Summary of Topics Covered in Today's Lecture

Some Applications involving Gravity -- Orbits and Escape Velocity

Newton was able to show mathematically that the shapes of orbits were *conic sections* -- circles, ellipses, parabolas, and hyperbolas. Deriving the shape or orbits is beyond the level of this course -- subject matter for an advanced course in mechanics -- but we can learn a few basic things about orbits and how their shape relates to the energy of the orbiting object.



In the picture is shown a small object of mass m orbiting a larger object of mass M. M's location is fixed -- it does not move. m is moving at speed v with respect to M and is located a distance r from M (as measured from center-of-mass to center-of-mass). In such a system there are two types of energy involved -- the Kinetic Energy and Potential Energy of m relative to M. The KE is simple:

$$KE = \frac{1}{2} mv^2$$

The Gravitational Potential Energy of m is fairly easy to figure, too. The potential at m's location due to the big mass M is

$$U_g = -GM/r$$

So m's PE is PE = m U_g or

$$PE = -GMm/r$$

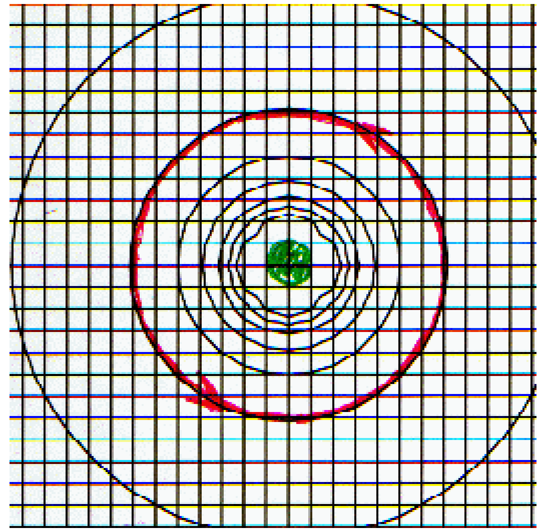
Therefore the total energy of the system is

$$E = KE + PE$$

$$E = \frac{1}{2} mv^2 - GMm/r$$

Circular Orbits

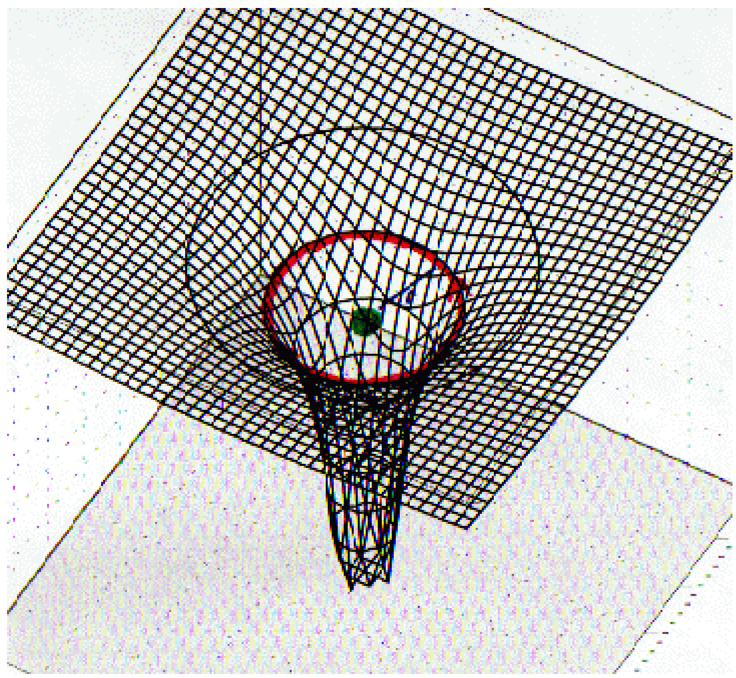
In a circular orbit the KE and PE are both constant and the total energy is negative. Essentially the orbit of m follows along an equipotential surface – which is a circle for a point mass.



In the figure at right we see a circular orbit (in red) around central mass (in green) plotted on an equipotential (or contour) plot of the potential function of the central mass.

Below that we have a surface plot of the potential function of the central mass showing the circular orbit.

Since *m* travels in a circular path, the physics of circular motion that you learned about in your first semester physics class is applicable to the orbit. Gravity provides the centripetal force between *M* & *m*:



The gravitational field created by *M* is

$$g = GM/r^2$$

and the gravitation force on *m* is given by

$$F_{grav} = m\ g = m(GM/r^2)$$

The force between *M* & *m* is

$$F_{grav} = GMm/r^2$$

Recall from Physics I that the force that makes m move in a circular path is the *centripetal force*, and the formula for centripetal force is

$$F_{centripetal} = mv^2/r$$

What is the centripetal force here? Gravity is the centripetal force! The force of gravity holds *m* in its circular path. Take gravity away and *m* flies off in a straight line.

$$F_{grav} = F_{centripetal}$$
$$GMm/r^2 = mv^2/r$$
$$GM/r = v^2 \qquad \textit{Cancel m, r}$$
$$v = (GM/r)^{1/2}$$

So the speed of an object in a circular orbit is

$$v_{circ} = \sqrt{\frac{GM}{r}}$$

The period of the orbit (T) is the time to complete one full orbit. The distance traveled in one orbit is the Circumference.

$$\text{Circumference} = v\,T$$
$$2\pi r = (GM/r)^{1/2}\,T$$
$$4\pi^2 r^2 = (GM/r)T^2$$
$$4\pi^2 r^3/GM = T^2$$

So the relationship between the period of the orbit and its radius is

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

The period of the orbit squared is proportional to its radius cubed. This is one the three laws of orbital motion discovered by Johannes Kepler in the 1600's. Kepler's Laws were a big step forward in the development of people's understanding how the universe works.

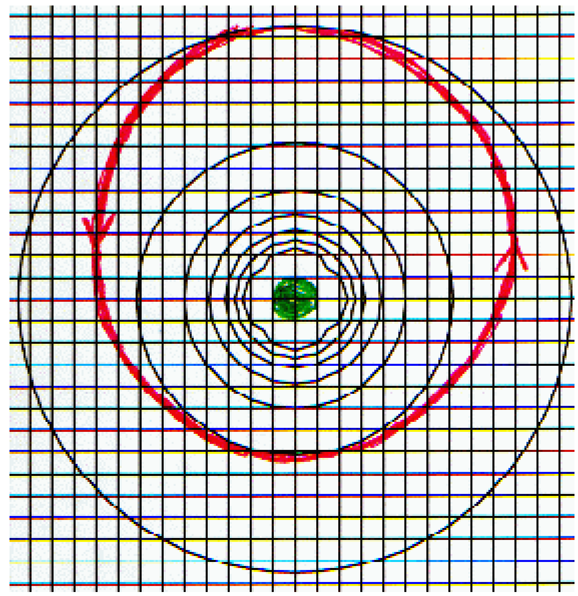
The total energy in the system is

$$E = KE + PE$$
$$= \tfrac{1}{2}\, mv^2 - GMm/r$$
$$= \tfrac{1}{2}\, m(GM/r) - GMm/r$$
$$= -\tfrac{1}{2}\, GMm/r$$

G, M, m, and r are all positive values. Thus the energy of an object in a circular orbit is said to be negative, or less than zero.

$$E_{circular} = -\tfrac{1}{2}\, GMm/r < 0$$

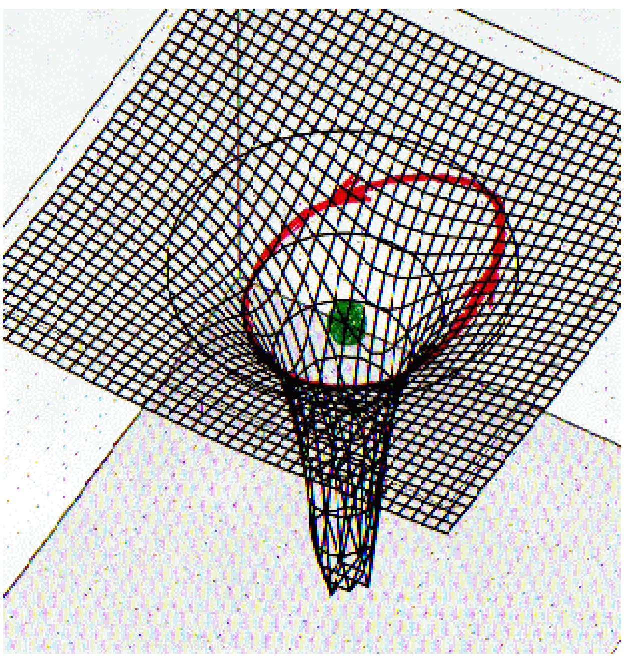
Elliptical Orbits



If an object is in a circular orbit and you give it a boost in speed it will start moving in an elliptical orbit.

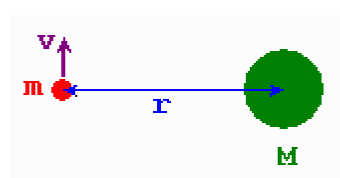
In an elliptical orbit the KE and PE vary – with KE reaching a maximum when *m* is closest to *M*, and PE reaching a maximum when *m* is furthest from *M*. The total energy is negative. Essentially the orbit of *m* is confined between two equipotential surfaces.

At right we see an elliptical orbit (in red) around central mass (in green) plotted on an equipotential (or contour) plot of the potential function of the central mass.



At left we have a surface plot of the potential function of the central mass showing the elliptical orbit.

Let's go back to "Newton's Mountain". Imagine the cannon has just enough powder to put the ball into a circular orbit. Then we add some more powder. This will mean the ball will actually have enough energy to move away from Earth a little bit before falling back, producing an elliptical orbit. Try this with the animation -- use 16,250 mph as the speed of the ball and you should see a circular orbit; raise the speed to about 17,000 mph and you will see the ball follow an elliptical path). Thus if we start out with the same set-up as a circular orbit and increase v a little bit we get an elliptical orbit. Increasing v means increasing energy, so an object in an elliptical orbit has greater energy than one in a circular orbit. However, as we shall see in a moment, the energy of an



object in an elliptical orbit is said to be negative, or less than zero.

$$E_{elliptical} = KE + PE$$

$$E_{elliptical} > E_{circular}$$

$$0 > E_{elliptical} > E_{circular}$$

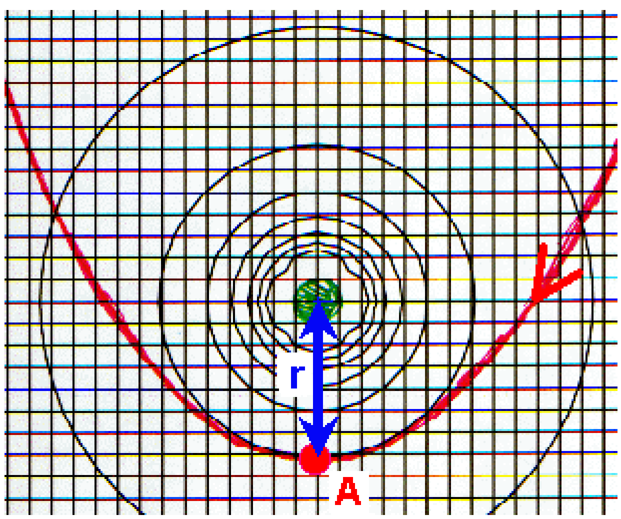
$$E_{circular} < E_{elliptical} < 0$$

so

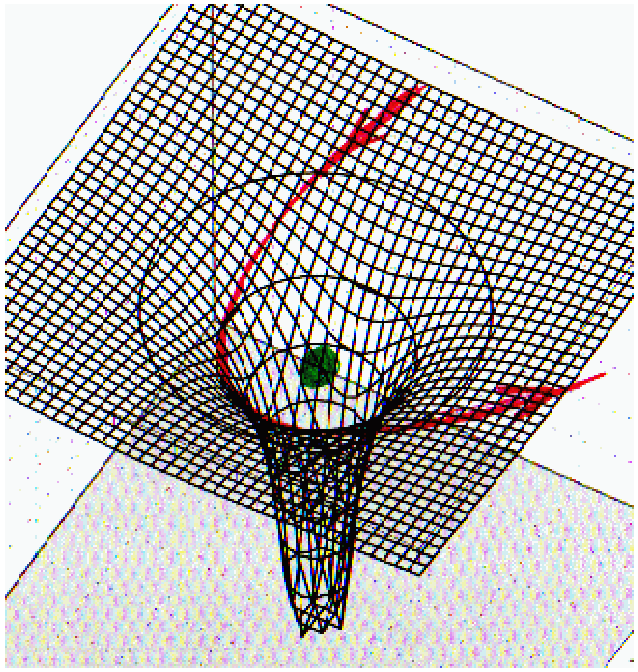
$$-\tfrac{1}{2}\, GMm/r < E_{elliptical} < 0$$

Parabolic Trajectories

In a parabolic trajectory the KE and PE vary - with KE reaching zero when the separation between *m* and *M* is infinite. The total energy is zero. Essentially the orbit of *m* goes out to infinity!



The figure above shows a parabolic trajectory (in red) around central mass (in green) plotted on a contour plot. The figure on the next page shows a surface plot of the potential function of the central mass showing the parabolic trajectory. The term “trajectory” is used because since m never comes back from infinity there is no “orbiting”. This is a one-time deal!



For m to get from its point of closest approach (at A in the figure) out to infinity, requires that

$$\mathbf{E_A} = \mathbf{E_\infty}$$
$$KE + PE = KE + PE$$
$$\tfrac{1}{2} mv^2 - GMm/r = 0 + GMm/\infty$$
$$\tfrac{1}{2} mv^2 - GMm/r = 0$$
$$\tfrac{1}{2} mv^2 = GMm/r$$
$$v^2 = 2GMm/r$$
$$v = (2GMm/r)^{1/2}$$

KE reaches zero when R is infinite

So the speed required to get to infinity from a distance r is

$$v_{escape} = \sqrt{\frac{2GM}{r}}$$

This is known as "escape velocity" because if m has this speed it can totally escape M's gravity – it will never return. If m has a speed of less than v_{escape}, even just a tiny amount less, it will be in an elliptical orbit and will eventually return to complete an orbit.

The total energy in the system is

$$E_{parabolic} = KE + PE = 0$$

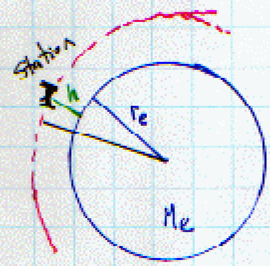
Thus the energy of an object in a parabolic trajectory is said to be zero. At object at rest a vast (infinite) distance away from a planet is unaffected by the planet. Thus it has no potential energy relative to the planet, and since it is not moving it has no kinetic energy, either. Relative to the planet it has zero energy. A parabolic trajectory will take an object to infinity, where it will be at rest. That’s why the energy of an object in a parabolic trajectory is said to be zero. The zero is a demarcation point. Objects with negative energy cannot escape the planet and are bound into an orbit.

Example Problem #1

If the Space Station orbits at an altitude of 170 miles, at what speed does it move and how long will it take to completely circle the Earth?

Solution:

I have to look up the mass and radius of Earth first. Then I can work the problem.



Radius of station orbit

$$r_s = r_e + h$$

$$r_e = 6.37 \times 10^6 \text{ m}$$

$$h = 170 \text{ miles} = 2.735 \times 10^5 \text{ m}$$

$$M_e = 5.98 \times 10^{24} \text{ kg}$$

$$r_s = 6.6435 \times 10^6 \text{ m}$$

$$V_{circ} = \sqrt{G \frac{M_e}{r_s}} = \sqrt{6.67 \times 10^{-11} \frac{N m^{\cancel{2}}}{kg^{\cancel{2}}} \frac{(5.98 \times 10^{24} \cancel{kg})}{6.6435 \times 10^6 \cancel{m}}}$$

$$V_{circ} = 7748.45 \text{ m/s} \quad \text{or} \quad 17000 \text{ mph}$$

Period

$$T = \frac{2\pi r_s}{v} = \frac{2\pi (6.6435 \times 10^6 \text{ m})}{7748.45 \text{ m/s}} = 5387.19 \text{ sec}$$

or 1.5 hours