DAY 4 – PHY 203/213

Summary of Topics Covered in Today's Lecture

BEGIN PHY 232 (CALCULUS PHYSICS) ONLY MATERIAL

More on Gradients

Now let's go beyond just looking, and show the math behind this.

$dU_g = -\mathbf{g}\cdot\mathbf{dr}$ and $dU_E = -\mathbf{E}\cdot\mathbf{dr}$ allow us to calculate potential difference if we know electric field, because we can integrate these equations.

$\Delta U_g = -\int \mathbf{g}\cdot\mathbf{dr}$ and $\Delta U_E = -\int \mathbf{E}\cdot\mathbf{dr}$

However, what is more often used is the derivative form of these equations, sort of

$-dU_g/\mathbf{dr} = \mathbf{g}$ and $-dU_E/\mathbf{dr} = \mathbf{E}$

But, this notation indicates a derivative taken with respect to a vector -- and actually is not real mathematical notation (a mathematician would probably not like the fact that I even wrote those blue equations down). A vector derivative is written with each component as a separate derivative:

$-(\delta U_g/\delta x\ \mathbf{i} + \delta U_g/\delta y\ \mathbf{j} + \delta U_g/\delta z\ \mathbf{k}) = \mathbf{g}$

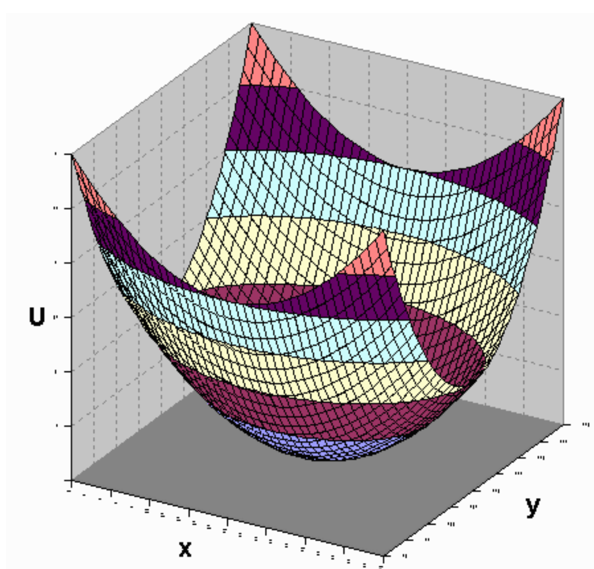
$-(\mathbf{i}\,\delta/\delta x + \mathbf{j}\,\delta/\delta y + \mathbf{k}\,\delta/\delta z)U_g = \mathbf{g}$

$-\nabla U_g = \mathbf{g}$

This ∇ is known as the gradient. The partial terms $\delta U_g/\delta x$, $\delta U_g/\delta y$, and $\delta U_g/\delta z$ are known as partial derivatives. What are these "partial derivatives"?

Consider a potential $U_g = x^2 + 2y^2$

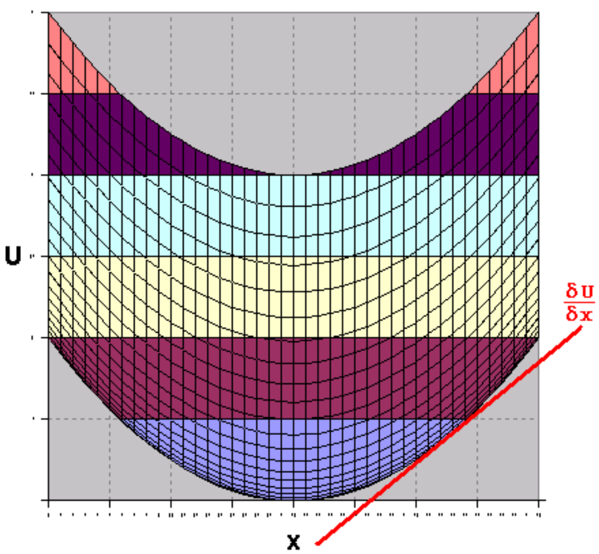
A graph of this potential is what we used when we were talking about gradients from a non-calculus perspective.



At right we again see a "surface plot" -- it that plots U as a

function of x and y for
$U_g = x^2 + 2y^2$.
Now we will use this plot of $U_g = x^2 + 2y^2$ to illustrate what is meant by a "partial derivative":

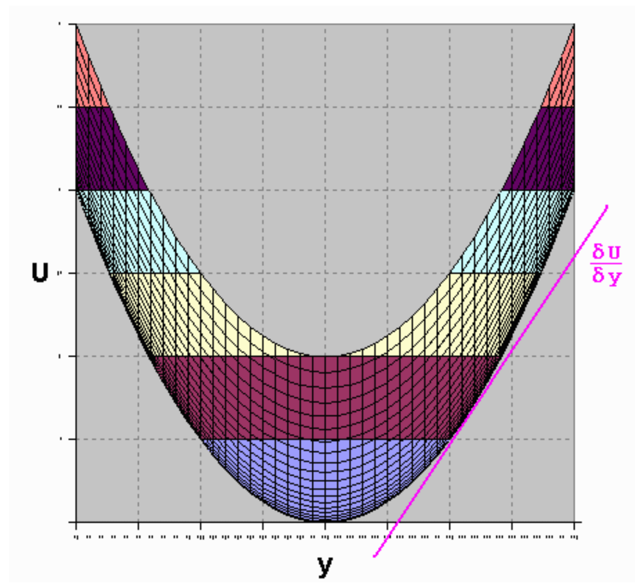
The partial derivative of U with respect to x is the slope a line tangent to the surface, in which the line runs only in the x direction (in the x-U plane). If we view our plot of $U_g = x^2 + 2y^2$ in the x-U plane we can see this:



This slope is independent of y, because y is constant in the x-U plane. Therefore, you take $\delta U_g/\delta x$ by taking the derivative of U with respect to x and treating y as a constant:

$\delta U_g/\delta x = \delta(x^2 + 2y^2)/\delta x = 2x + 0 = 2x$

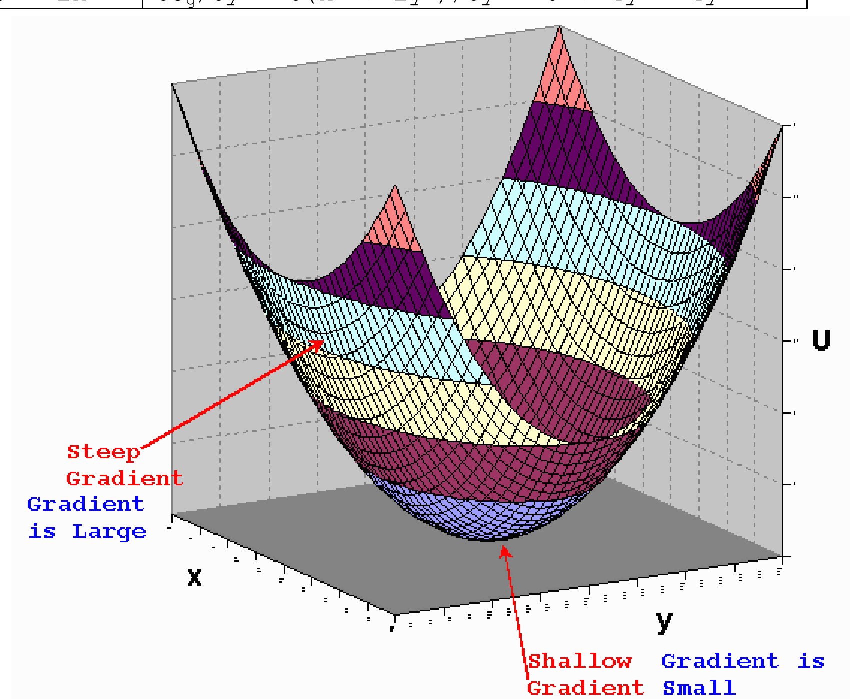
The partial derivative of U with respect to y is the slope a line tangent to the surface, in which the line runs only in the y direction (in the y-U plane). If we view our plot of $U_g = x^2 + 2y^2$ in the y-U plane we can see this:



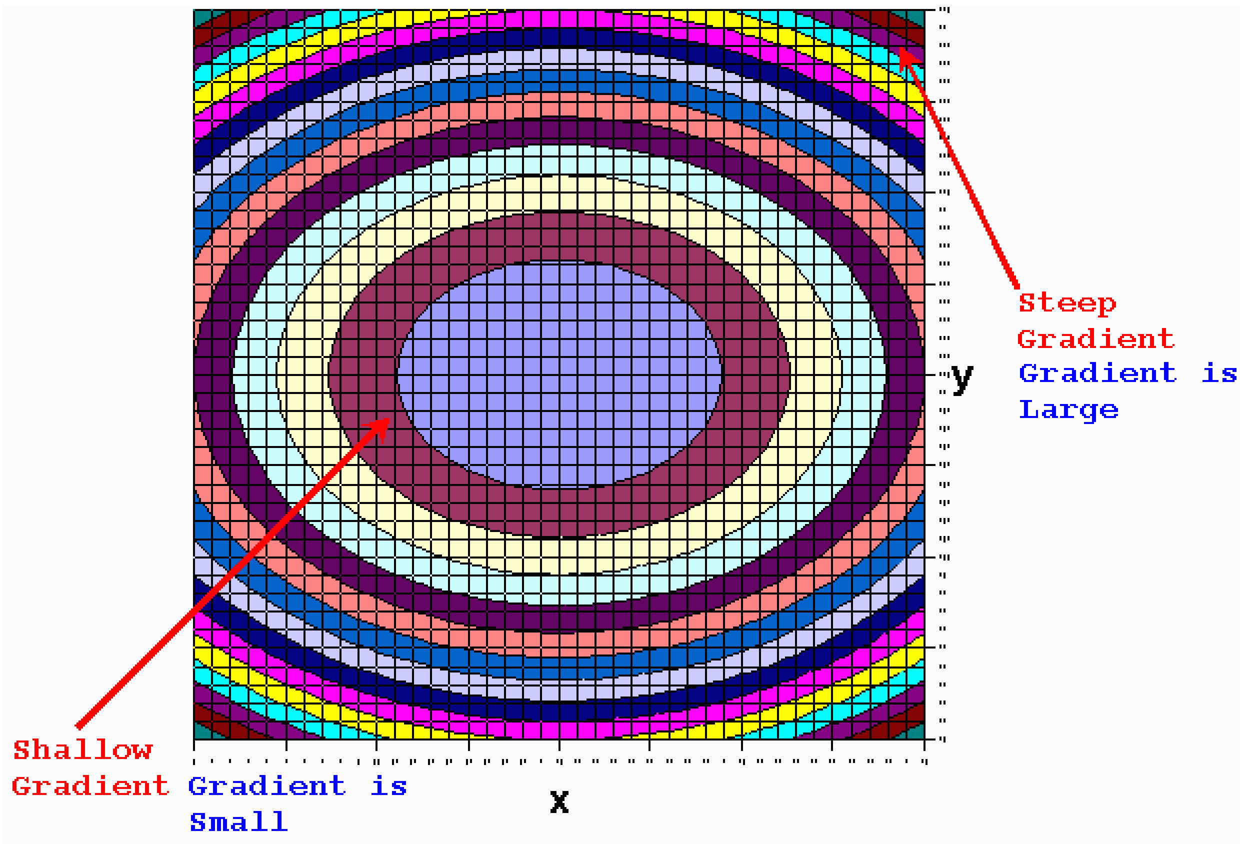
This slope is independent of x, because x is constant in the y-U plane. Therefore, you take $\delta U_g/\delta y$ by taking the derivative of U with respect to y and treating x as a constant:

$\delta U_g/\delta y = \delta(x^2 + 2y^2)/\delta y = 0 + 4y = 4y$

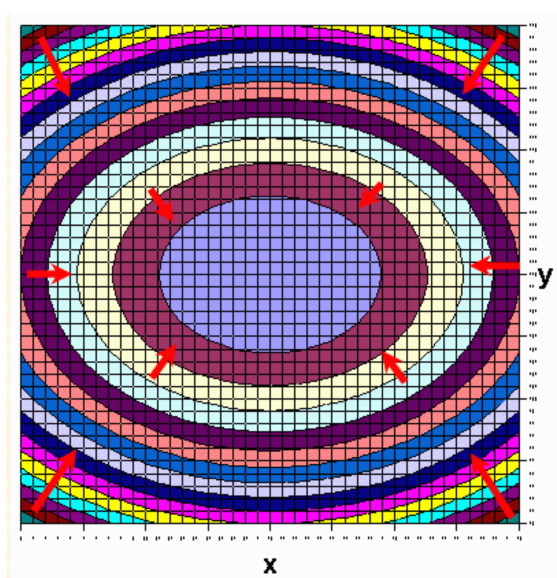
The gradient combines these two into a slope of the overall surface in the "up hill" direction. The steeper the slope of the surface, the larger the gradient.



Let's look at this in terms of equipotential surfaces. The equipotential surface plot of $U_g = x^2 + 2y^2$ is shown again below. Note that far from the "bottom" of the plot, where the gradient is steep, equipotential surfaces are closely spaced. Near the "bottom" of the plot, where the gradient is shallow, equipotential surfaces are widely spaced.



And while the gradient vector points "uphill", the field is the negative of the gradient, so the field vector points "downhill", and of course perpendicular to the equipotential surfaces (as we learned earlier).



So again let's draw in some field vectors that point downhill, that are perpendicular to the equipotential surfaces, and that are larger where the surfaces are closely spaced and smaller where the surfaces are widely spaced.

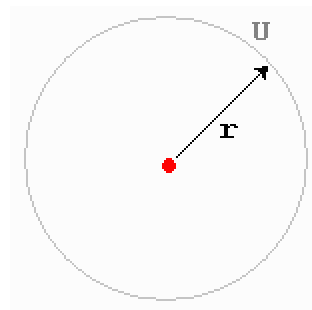
END PHY 232 MATERIAL

The Potential of a Point

A single point should have a potential that is

(a) radially symmetric.
(b) decreasing in effect with distance from the point, to the extent that at a large distance from the point the effect of the point is zero.

The function 1/r is the simplest function that satisfies these. It also happens to agree with experiments regarding the potentials of point masses and point charges. The potential at a distance r from a point charge of q is



$U_E = kq/r$

The potential at a distance r from a point mass of m is

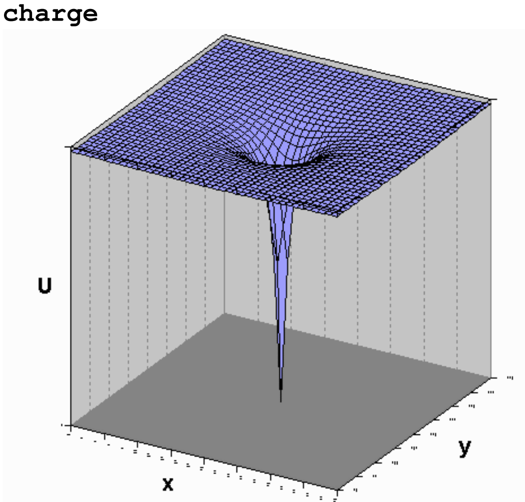
$U_g = -Gm/r$

The constant k is referred to as the Coulomb Constant and has been determined by experiment to be $k = 9 \times 10^9\ Nm^2/C^2$

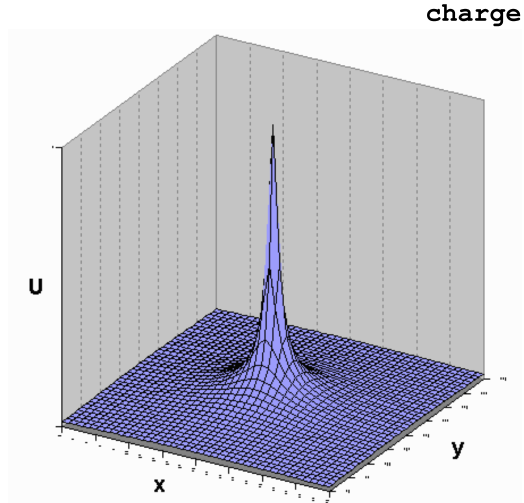
The constant G is referred to as the Universal Gravitational Constant and has been determined by experiment to be $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

Plots of the potentials of point objects look like this:

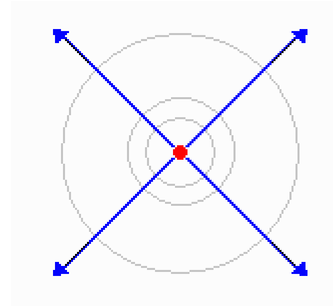
Gravitational potential of a mass, or electric potential of a negative charge



Electric potential of a positive charge



The fields created by point charges and masses can be found by taking the gradient of these potential functions. They can be visualized by drawing equipotential surfaces, and then drawing in field lines in the direction of the field.



Field lines and equipotential surfaces for a positive point charge.

Example Problem #1 (PHY 232 Only)

An electric potential is given by $U_E = (2x + y + xy^2)^2$. Determine an equation for the Electric Field. Calculate the magnitude of the **E** field at the following points: (0,0); (1,1).

Solution:

$\mathbf{E} = -\nabla U_E = -(\mathbf{i}\,\delta/\delta x + \mathbf{j}\,\delta/\delta y + \mathbf{k}\,\delta/\delta z)U_E = -(\mathbf{i}\delta U_E/\delta x + \mathbf{j}\,\delta U_E/\delta y + \mathbf{k}\,\delta U_E/\delta z)$

$\delta U_E/\delta x = 2(2x + y + xy^2)(2 + y^2)$ Treat y as a constant and differentiate with respect to x.

$\delta U_E/\delta y = 2(2x + y + xy^2)(1 + 2xy)$ Treat y as a constant and differentiate with respect to x.

$\mathbf{E} = -2(2x + y + xy^2)(2 + y^2)\mathbf{i} - 2(2x + y + xy^2)(1 + 2xy)\mathbf{j}$
$\mathbf{E} = -2(2x + y + xy^2)[(2 + y^2)\mathbf{i} + (1 + 2xy)\mathbf{j}]$

At (0,0) I get

$\mathbf{E} = -2(2(0) + 0 + (0)0^2)[(2 + 0^2)\mathbf{i} + (1 + 2(0)(0))\mathbf{j}] = 0$

At (1,1) I get

$\mathbf{E} = -2(2(1) + 1 + (1)1^2)[(2 + 1^2)\mathbf{i} + (1 + 2(1)(1))\mathbf{j}]$
$\mathbf{E} = -2(4)[(3)\mathbf{i} + (3)\mathbf{j}]$
$\mathbf{E} = -8[(3)\mathbf{i} + (3)\mathbf{j}]$
$\mathbf{E} = -24\mathbf{i} - 24\mathbf{j}$

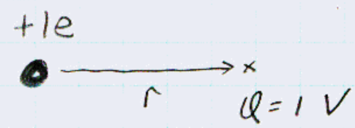
$E_x = -24,\ E_y = -24 \quad E = [(-24)^2 + (-24)^2]^{(1/2)} = 33.94$

Example Problem #2

How far away from an isolated proton will the potential be 1.0 V?

Solution:

$$V = k\frac{q}{r}$$



$$k = 9\times10^{9}\ \frac{Nm^2}{C^2} \qquad q = 1e = 1.6\times10^{-19}\ C$$

$$1\ V = 9\times10^{9}\ \frac{Nm^2}{C^{\cancel{2}}}\left(\frac{1.6\times10^{-19}\ \cancel{C}}{r}\right)$$ Coulombs partially cancel.

$$1\ \frac{J}{\cancel{C}} = 1.44\times10^{-9}\ \frac{Nm^2}{\cancel{C}}\left(\frac{1}{r}\right)$$ Coulombs cancel

$$r = \frac{1.44\times10^{-9}\ Nm^2}{1\ J}$$

$$= \frac{1.44\times10^{-9}\ \cancel{Nm^{2}}}{1\ \cancel{Nm}}$$

$$r = 1.44\times10^{-9}\ m$$

The potential will be 1 V at a distance of 1.44 nm from an isolated proton.

Example Problem #3 **(PHY 232 Only)**

Newton's Universal Law of Gravitation was supposedly inspired by an apple falling from a tree in front of Newton while he was tending sheep and very bored. The law says that two masses, m_1 & m_2, separated by a distance r, are attracted by a force

$$F = G \frac{m_1 m_2}{r^2}$$

Derive this equation from our definition of potential for a point mass. (HINT -- Take the gradient, then use F = m g.)

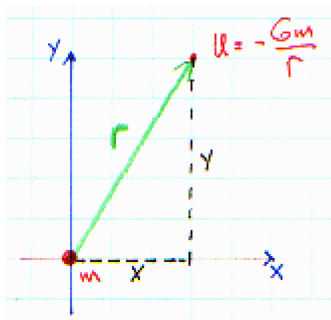
Calculate the force between the Earth and Moon. Are the Earth & Moon point masses?

Solution:

Starting with U_g for a point mass...

$$U_g = -\frac{Gm}{r}$$

$$U_g = \frac{-Gm}{\sqrt{x^2+y^2}}$$



$$r = (x^2+y^2)^{\frac{1}{2}}$$

...Find g via the gradient!

$$\vec{g} = -\nabla U_g$$

$$= -\left(\hat{i}\,\frac{dU_g}{dx} + \hat{j}\,\frac{dU_g}{dy}\right)$$

OK, let's do the x partial derivative first...

$$\frac{dU_g}{dx} = -Gm\frac{d}{dx}(x^2+y^2)^{-\frac{1}{2}}$$

$$= +\frac{1}{2}Gm(x^2+y^2)^{-3/2}(2x+0)$$

$$= \frac{+Gm\,x}{(x^2+y^2)^{3/2}}$$

$$= +\frac{Gmx}{r^3}$$

Doing the y partial derivative is just like the x partial derivative, so I'm just going to write down the same answer, only with y instead of x.

$$\frac{dU_g}{dy} = +\frac{Gmy}{r^3}$$ This is just like $\frac{dU_g}{dx}$!

Now I'll put it all together…

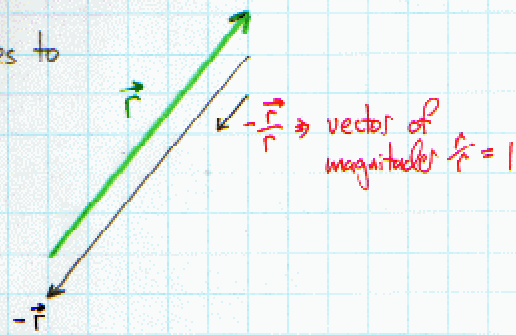
$$\vec{g} = -\hat{\imath}\frac{dU_g}{dx} - \hat{\jmath}\frac{dU_g}{dy}$$

$$= -\frac{Gmx}{r^3}\hat{\imath} - \frac{Gmy}{r^3}\hat{\jmath}$$

$$= -\frac{Gm}{r^3}\left(x\hat{\imath} + y\hat{\jmath}\right)$$ but $\vec{r} = x\hat{\imath} + y\hat{\jmath}$

$$\vec{g} = -\frac{Gm}{r^3}\vec{r} = \frac{Gm}{r^2}\left(-\frac{\vec{r}}{r}\right)$$

$-\frac{\vec{r}}{r}$ reduces to 1



$$\vec{g} = \frac{Gm}{r^2}\left(-\frac{\vec{r}}{r}\right)$$

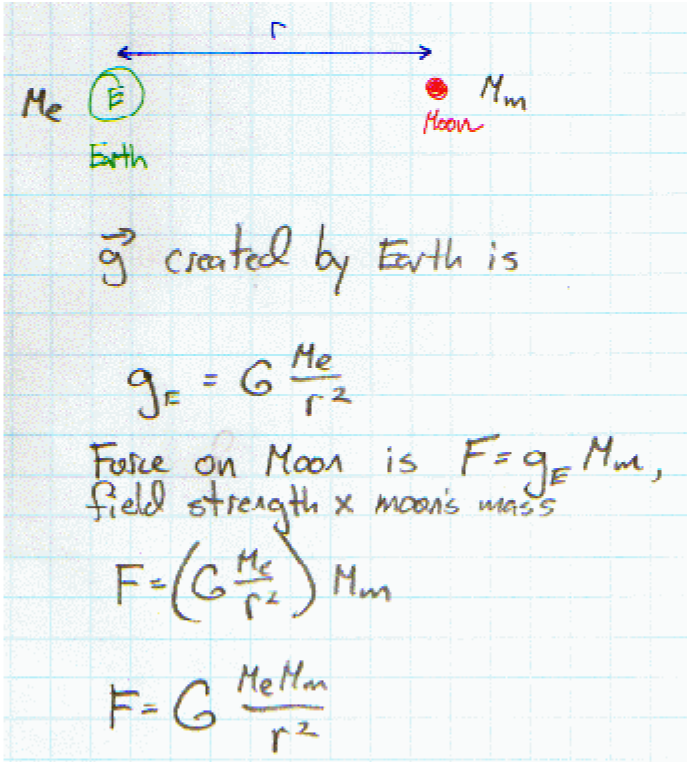
$$\vec{g} = \frac{Gm}{r^2}$$ pointing toward m

The magnitude of g is given by

$$g = G\frac{M}{r^2}$$

and the direction is toward Earth.

Now we'll use this equation and the idea that F = mg to get the Universal Law of Gravitation:



(**Here's a little side note--**
The equivalent law for electricity is called Coulomb's Law. Coulombs law says that the force between two charges q1 and q2 separated by a distance r is

$$F = k\frac{q_1 q_2}{r^2}$$

directed along a line separating the two charges.)

Now let's find the force between the Earth and the Moon.

Earth mass $M_e = 5.98 \times 10^{24}\ \text{kg}$

Moon mass $M_m = 7.36 \times 10^{22}\ \text{kg}$

separation $r = 3.84 \times 10^{8}\ \text{m}$

Values looked up in a book

$$F = G\frac{M_e M_m}{r^2}$$

$$= 6.67 \times 10^{-11}\ \frac{N\cancel{m^2}}{\cancel{kg^2}}\ \frac{(5.98 \times 10^{24}\ \cancel{kg})(7.36 \times 10^{22}\ \cancel{kg})}{(3.84 \times 10^{8}\ \cancel{m})^2}$$

$$\boxed{F = 2 \times 10^{20}\ N}$$