DAY 16

Summary of Topics Covered in Today's Lecture

Damped Oscillations

In a Simple Harmonic Oscillator energy oscillates entirely between Kinetic Energy of the mass and Elastic Potential Energy stored in the spring. The total Energy in the system does not change, so a SHO will oscillate forever.

In most oscillators, however, there is friction. A simple friction force is that produced by viscous fluids, where the drag force on an object moving with velocity **v** is given by $\mathbf{F}_{drag} = -b\,\mathbf{v}$. If there is a drag force in the oscillator then we can find the equation of motion of the "damped" oscillator by using Newton's Second Law of motion:
$\Sigma\mathbf{F} = m\,\mathbf{a}$:

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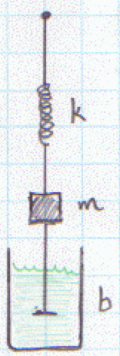

$$\Sigma F = ma$$

Spring force + Damp drag force = ma

$$-kx - bv = ma$$

$$0 = ma + bv + kx$$

$$0 = a + \frac{b}{m}v + \frac{k}{m}x$$

We need more math than just algebra to be able to <u>prove</u> what the solution for this motion is, so we'll just state it:

The solution to this is

$x = A\cos(\omega t + \phi)$ as before, except that the amplitude of oscillation decays over time:

$$A = A_0 e^{-\frac{b}{2m}t}$$

Also, the frequency is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

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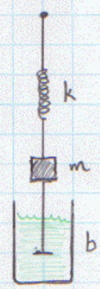

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Spring force + Damp drag force = ma

$$-kx - bv = ma$$

$$0 = ma + bv + kx$$

$$0 = a + \frac{b}{m}v + \frac{k}{m}x$$

$$a = \frac{d^2x}{dt^2} \qquad v = \frac{dx}{dt}$$

So the differential equation for the damped oscillator is

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

The solution to this is

$x = A\cos(\omega t + \phi)$ as before, except that the amplitude of oscillation decays over time:

$$A = A_0 e^{-\frac{b}{2m}t}$$

Also, the frequency is

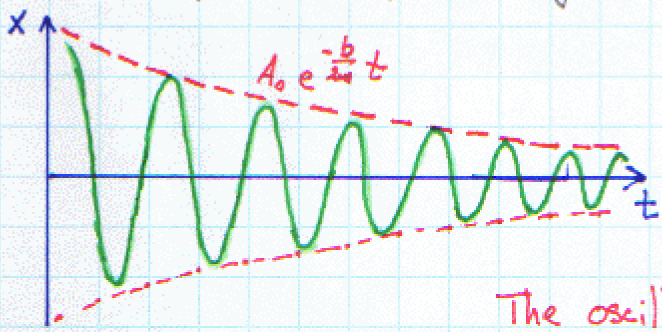
$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

The two key terms here are

$$\frac{k}{m} \quad \text{and} \quad \left(\frac{b}{2m}\right)^2$$

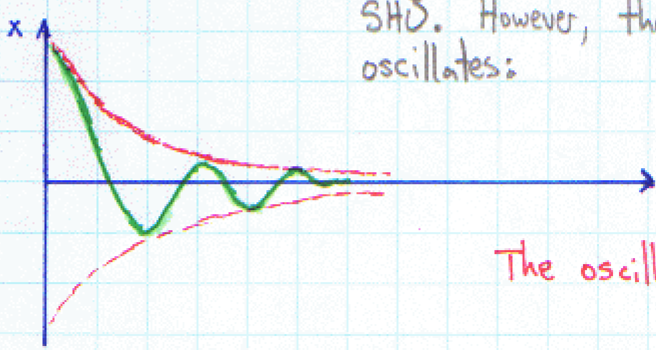
If $\frac{k}{m} \gg \left(\frac{b}{2m}\right)^2$ then $\omega \simeq \sqrt{\frac{k}{m}}$ and

the system behaves must like an undamped SHO - only the amplitude gradually decays:



The oscillator is "lightly damped"

If $\frac{k}{m} > \left(\frac{b}{2m}\right)^2$ then the ω frequency is no longer the same as an undamped SHO. However, the system still oscillates:



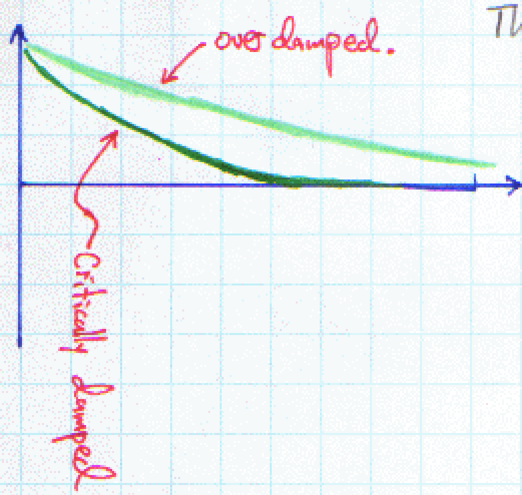
The oscillator is "underdamped"

If $\frac{k}{m} = \left(\frac{b}{2m}\right)^2$ then $\omega = 0$ and the system does not oscillate.

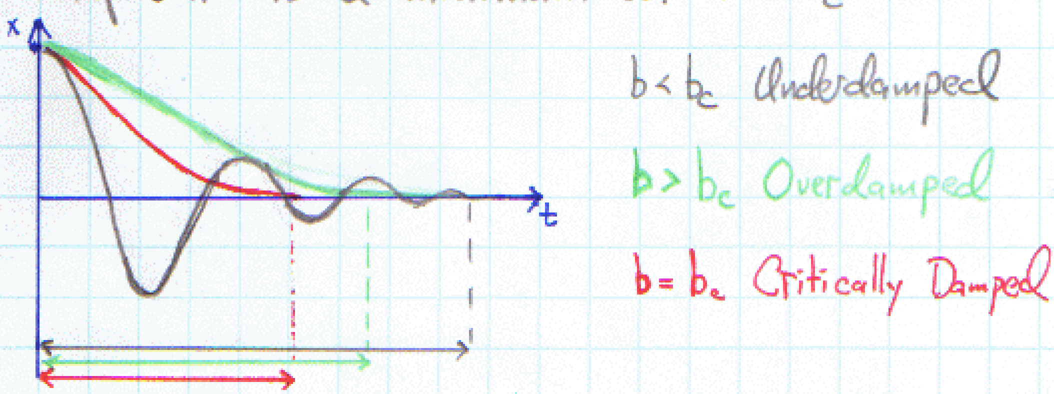
The oscillator is "critically damped" and the damping coefficient is $b_c = 2\sqrt{mk}$. The oscillator moves toward equilibrium without oscillating.

$$b_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{mk}$$

If $\frac{k}{m} < \left(\frac{b}{2m}\right)^2$ the ω is a complex number. The mass moves toward equilibrium as in the critically damped case, but takes longer to do so. The oscillator is "over damped."



The time required for the oscillator to lose most of its energy to friction – i.e. for the oscillations to "damp out" is a minimum for $b = b_c$



The time for oscillations to "damp out" is a minimum for the critically damped case.

Note that in this treatment we are using the cosine function instead of the using the sine function like we did last class. This is just because the starting point is more convenient to show the exponential part of damped oscillations. Both sine & cosine are "sinusoidal" – they are exactly the same in their shape. Either can be used.

The Driven Harmonic Oscillator and Resonance

Now let's consider the case where we add something to our damped oscillator that drives it to oscillate. We can produce an external driving force by having the oscillator hanging from a string that is pulled back and forth by a rotating wheel.

Lets suppose that our driver produces a sinusoidal external driving force of $F_D = F_{ext}\cos(\omega_D t)$. Now we add this into our Newton's Second Law equation to get an equation for this system:

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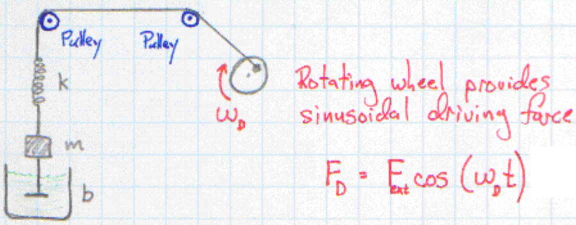

$$F_D = F_{ext}\cos(\omega_D t)$$

$$\Sigma F = ma$$

Driving force + Spring force + Damp Drag force = ma

$$F_{ext}\cos(\omega_D t) - kx - bv = ma$$

Again, we need more math than just algebra to be able to prove what the solution for this motion is, so we'll just state it:

$$x = A\cos(\omega_D t + \phi)$$

ω_D is the frequency of the driving force.

$$A = \frac{F_{ext}/m}{\sqrt{\left(\omega_D^2 - \frac{k}{m}\right)^2 + \left(\frac{b\omega_D}{m}\right)^2}}$$

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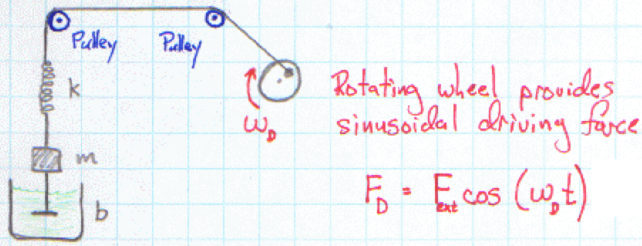


$$F_D = F_{ext}\cos(\omega_D t)$$

$$\Sigma F = ma$$

Driving force + Spring force + Damp Drag force = ma

$$F_{ext}\cos(\omega_D t) - kx - bv = ma$$

$$\frac{F_{ext}}{m}\cos(\omega_D t + \phi) = \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x$$

The solution to this differential equation is

$$x = A\cos(\omega_D t + \phi)$$

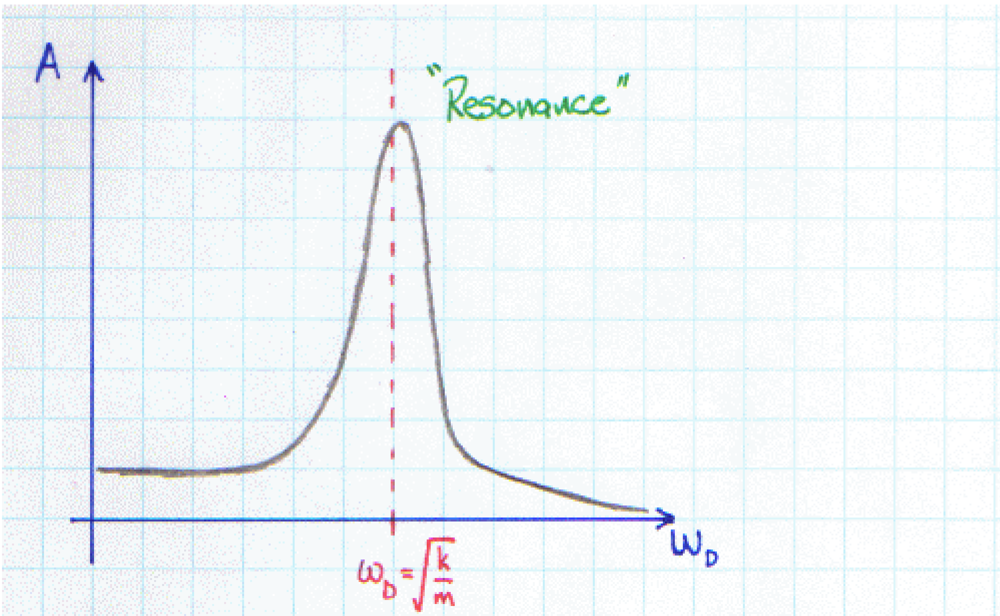
ω_D is the frequency of the driving force.

$$A = \frac{F_{ext}/m}{\sqrt{\left(\omega_D^2 - \frac{k}{m}\right)^2 + \left(\frac{b\omega_D}{m}\right)^2}}$$

A graph of the equation

$$A = \frac{F_{ext}/m}{\sqrt{\left(\omega_D^2 - \frac{k}{m}\right)^2 + \left(\frac{b\omega_D}{m}\right)^2}}$$

in which we plot A vs. ω_D looks something like this (for the underdamped case):



When the driving frequency matches the "natural frequency" of the oscillator, the amplitude of oscillation increases dramatically.
This phenomenon is known as "resonance".

Because so many things act like harmonic oscillators (because so many materials are elastic), resonant behavior shows up everywhere. From the vibrations of bridges and buildings to musical instruments to that odd rattle in your car that only appears at a particular speed, resonance is everywhere.

The Steady-State Picture

These pictures of oscillators involve are valid only for the "long-term" or "steady-state" case. If you start driving any oscillator, it will eventually settle down into the kinds of motion we've discussed here. However, when you first start an oscillator moving, it may do a lot of weird things before settling down into its long-term motion. The initial behavior of oscillators is called "transient" behavior and is beyond what we can do in this course.

Example Problem #1

An oscillator with mass 10 kg and undamped frequency of 2 Hz is driven by a driving force of amplitude 5 N.

(a) Determine the spring constant of the oscillator.
(b) Determine what damping coefficient is require for critical damping.
(c) Make plots of Amplitude vs. driving frequency (in Hz) if the oscillator is lightly damped, underdamped, critically damped, and overdamped. Use EXCEL and plot from f_D = 0 to f_D = 4 Hz

Solution:

$m = 10\ \text{kg}$

First let's find k

$f_o = 2\ \text{Hz}$ undamped freq

$\omega_o = 2\pi f = 4\pi\ \frac{1}{s} = 12.5664\ \frac{1}{s}$

$\omega_o = \sqrt{\frac{k}{m}} \qquad \omega_o^2 = \frac{k}{m} \qquad k = m\,\omega_o^2$

$= 10\ \text{kg}\left(12.5664\ \frac{1}{s}\right)^2$

$\approx 1579.14\ \text{kg}/s^2$

Do the units work out?

$\frac{N}{m} \rightarrow \frac{\text{kg}\,\cancel{m}/s^2}{\cancel{m}} \rightarrow \text{kg}/s^2$ units OK ✓

So k = 1579 N/m.

Now I can use k & m to find the critical damping coefficient:

$$b_c = 2\sqrt{mk} = 2\sqrt{10\text{ kg}\left(1579.14\text{ kg}/s^2\right)}$$
$$= 251.327\text{ kg}/s$$

b_C = 251 kg/s.

I'm going to use EXCEL to do all this. I'll substitute $2\pi f$ for ω in my Amplitude equation, and use the external driving force, k, and b in the equation:

$$F_{ext} = 5\text{ N}$$
$$A = \frac{F_{ext}/m}{\sqrt{\left((2\pi f_D)^2 - k/m\right)^2 + \left(\frac{b\,2\pi f_D}{m}\right)^2}}$$

First I'll do the critically damped case.

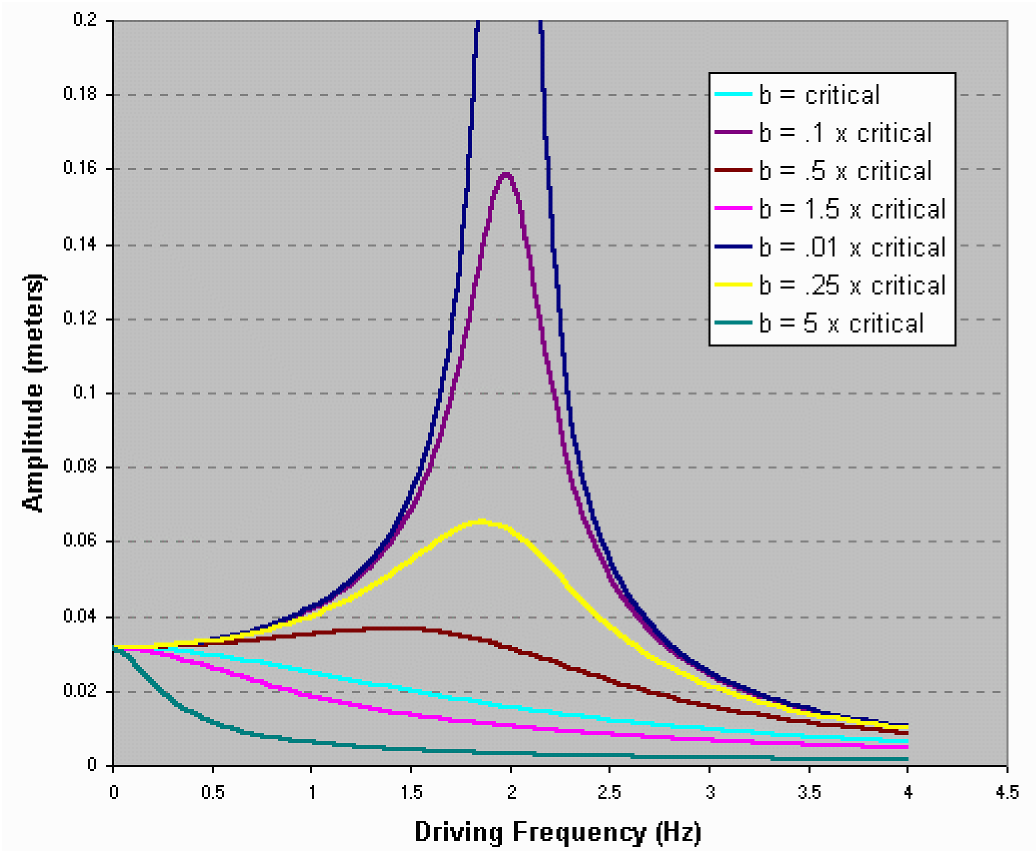
b_C = 251.327

A = (5/10)*((2*3.14159*f)^2 – 1579.14/10)^2 + (251.327*2*3.14159*f/10)^2)^-0.5

That's what is going into EXCEL.

For lightly damped I'll use b = .1 b_C.
For underdamped I'll use b = .5 b_C.
For overdamped I'll use b = 1.5 b_C.
I arrived at these values by trial and error.

Then it was so easy to make graphs with EXCEL that I added plots for b = .01 b_C, b = .25 b_C, and b = 5 b_C, just for kicks. Here are the results:



Note that at very low frequencies the amplitudes are all the same. That's because if the oscillator is moving slowly enough the drag force is negligible.