DAY 14 – Note that there are some "animations" on this page that are marked with a red arrow. See the supplemental page to see these animations in action.

Summary of Topics Covered in Today's Lecture

Oscillations

Any cyclic back-and-forth action that results in no net motion is an oscillation. Oscillations tend to be characterized by their Period (the time to complete one cycle of the oscillation, usually abbreviated with a T) and their Amplitude (the size of the oscillation, usually abbreviated with an A).

The frequency of the oscillation is the number of cycles of oscillation completed per second:

f = # cycles/time

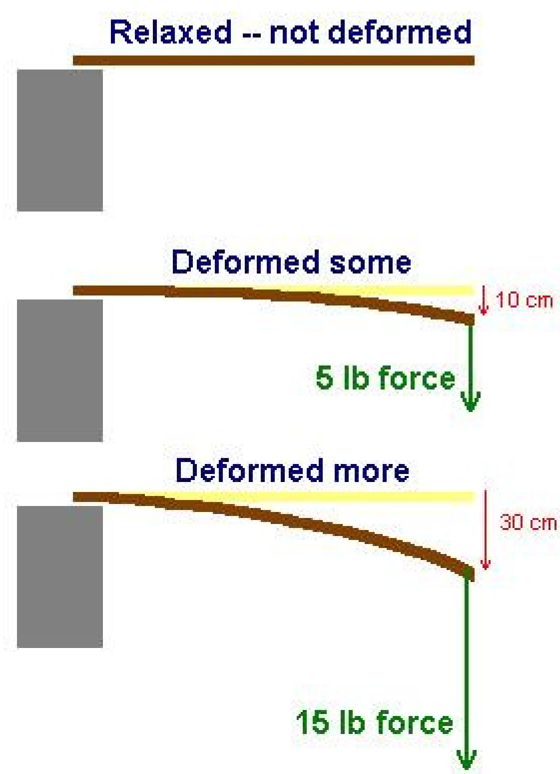
f = 1/T

The SI unit of frequency is the cycles/second or *Hertz* (Hz).

Hooke's Law & the Simple Harmonic Oscillator

Of particular interest is the Simple Harmonic Oscillator (SHO). A SHO is an oscillator that oscillates due to a *Hooke's Law* force. Hooke's Law is

$$F = -kx$$



where F is the amount of force required to deform the object, k is the stiffness or *Spring Constant* of the object, and x is the amount the object is deformed.

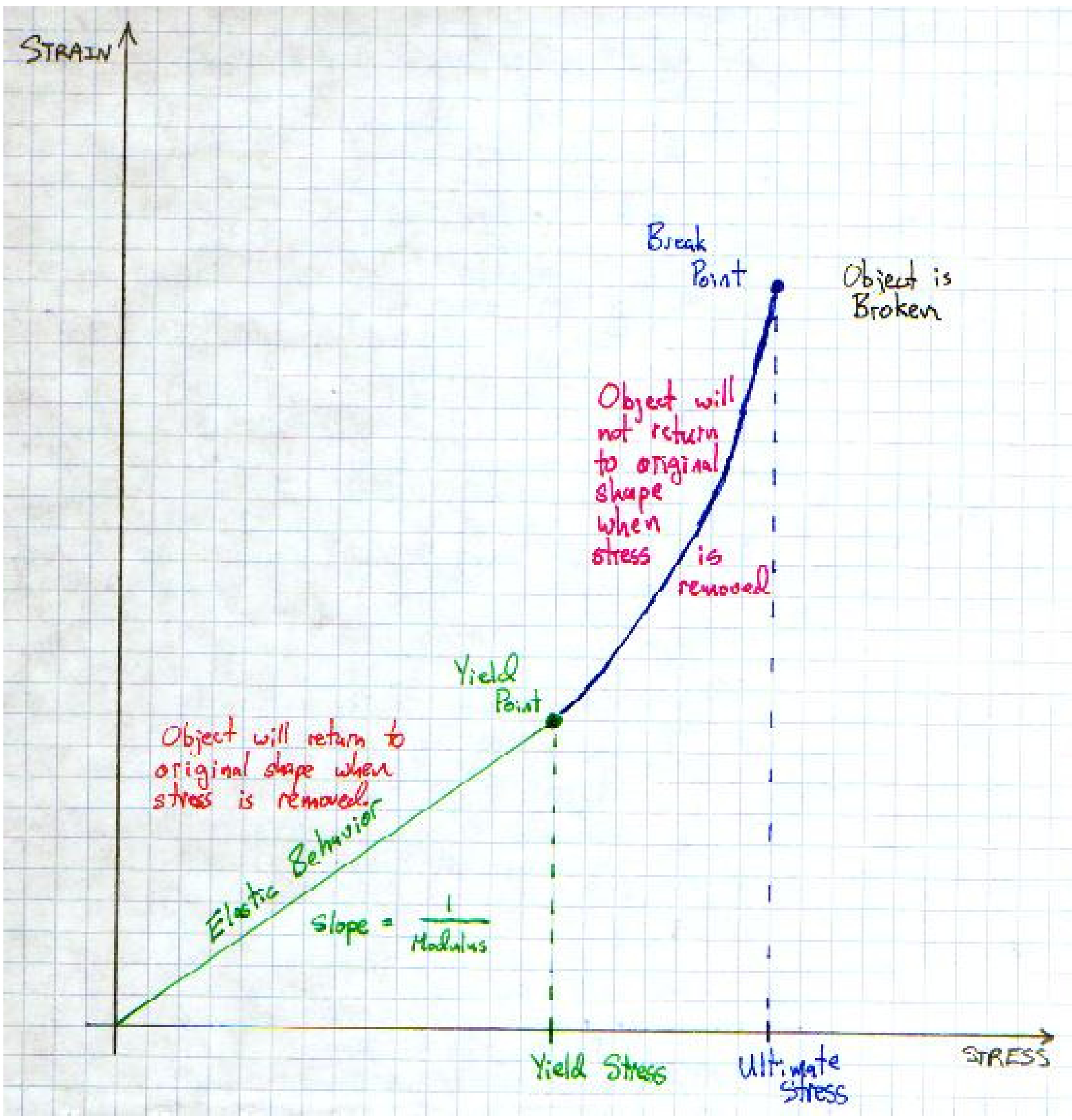
Hooke's Law forces are important because they are so common. Hooke's Law applies any time that the force required to deform an object (like a spring or a flexible stick) is proportional to the amount the object is deformed. This is true for any elastic object. If you recall from

Physics I, the ratio between stress and strain in an elastic material is linear:

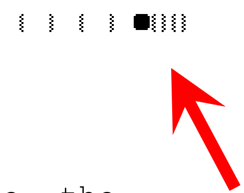
Stress = Modulus x Strain

Stress is that which causes deformation (akin to *F* in Hooke's Law). Strain is a measure of amount of deformation (akin to *x* in Hooke's Law). Modulus relates the two -- akin to *k* in Hooke's Law. In fact, the concepts of Modulus and *k* (Spring Constant) are closely related, except that spring constant applies to a particular object (i.e. a *steel spring* 10 cm long, 2 cm wide, and 2 mm thick), whereas Modulus applies to a material (i.e. *steel*).

Materials only show elastic behavior within limits – those limits being that the strain the object is subjected to is small enough that the object will return to its original shape when the deforming stress is released. Deform any object too much and it will no longer return to its original shape – it has been deformed past its yield point. Keep deforming an object past the yield point and it tends to deform more dramatically. Eventually it will break.



So as long as an object is not stressed past its Yield Point, its behavior is elastic and Hooke's Law forces apply. If the object is deformed and released, it will oscillate – it will be a Simple Harmonic Oscillator, and its motion will look something like the animation at right.



The Motion of a Simple Harmonic Oscillator

If an object of mass m is moving under a Hooke's Law force, the sum of the forces acting on it is

$$\sum \mathbf{F} = m\mathbf{a} = -k\mathbf{x}$$

$$m\mathbf{a} = -k\mathbf{x}$$

$$m\mathbf{a} + k\mathbf{x} = 0$$

$$\mathbf{a} + \frac{k}{m}\mathbf{x} = 0$$

(The boldface type indicates vector quantities.)

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We can't prove this using algebra, but the motion this equation implies is sinusoidal – back and forth like a sine wave. A good look at a Simple Harmonic Oscillator will confirm this. A sine wave has the equation

$$\mathbf{x} = \mathbf{A}\sin(\omega t + \phi)$$

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Recall that acceleration is the derivative of velocity with respect to time, and the second derivative of position with respect to time:

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}\left(\frac{d}{dt}\mathbf{x}\right) = \frac{d^2}{dt^2}\mathbf{x}$$

so we now have

$$\frac{d^2}{dt^2}\mathbf{x} + \frac{k}{m}\mathbf{x} = 0$$

Below is a plot of what this equation looks like for different values of A, ω, and ϕ.

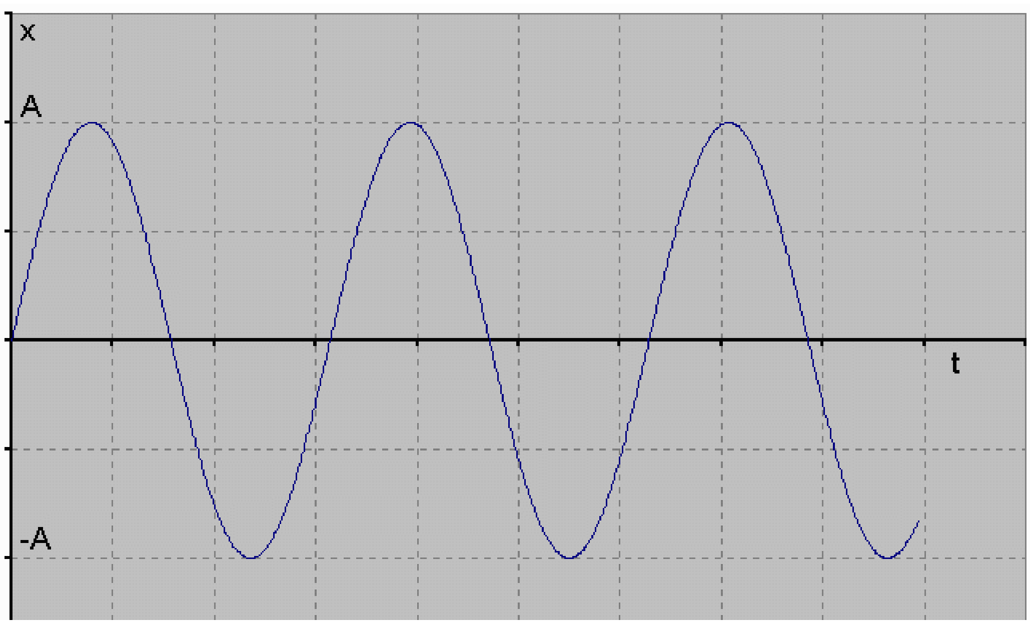
This is what is known as a *differential equation* in **x**. Now, we don't know how to solve a differential equation, but we can guess at a solution and check to see if it works by substituting it into this equation. A solution would have to be an equation for **x** that gives back something like **x** when you take its derivative twice. That sounds like a sine or cosine function.

Let's try

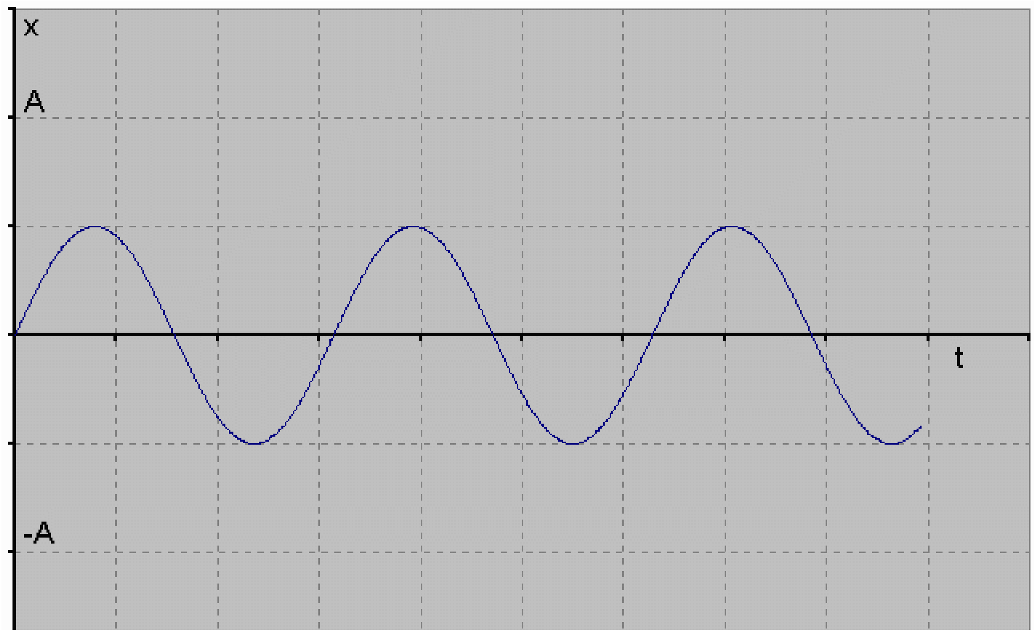
$$\mathbf{x} = \mathbf{A}\sin(\omega t + \phi)$$

Below is a plot of what this equation looks like for different values of A, ω, and ϕ.

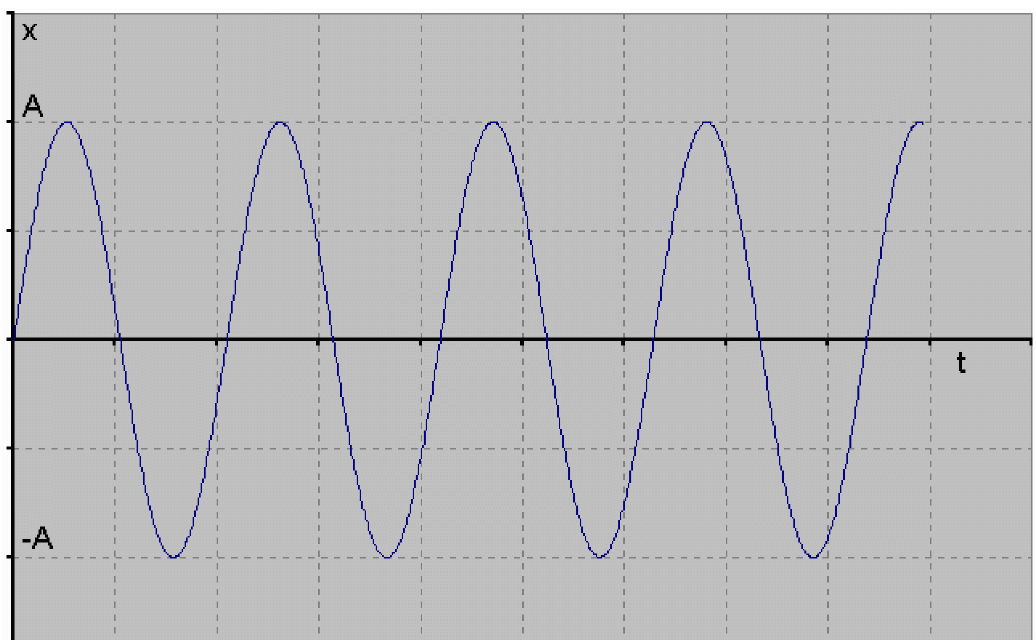
A = 10
ω = 2
ϕ = 0

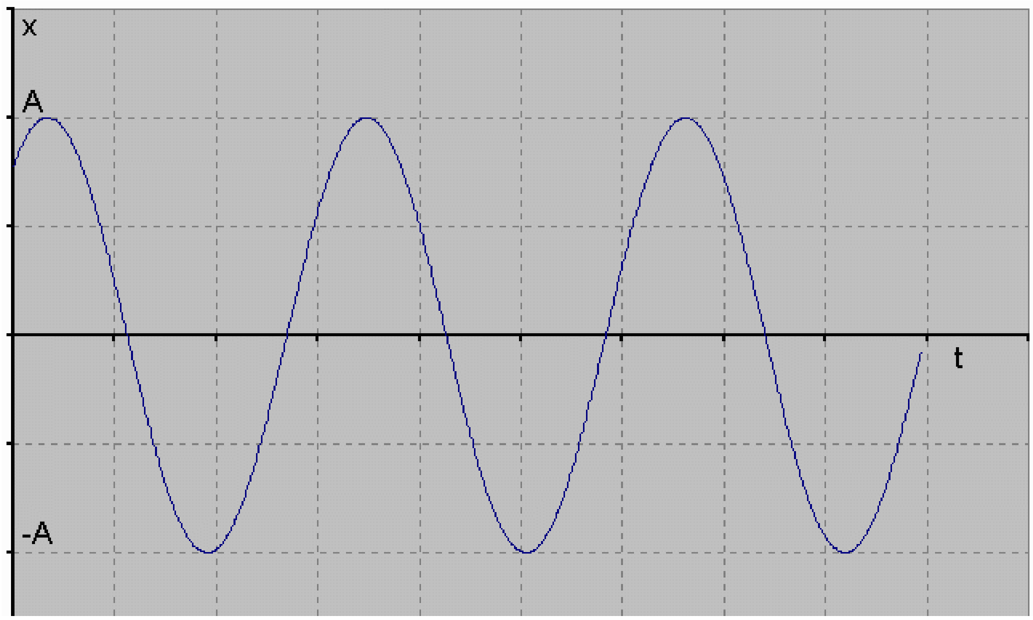


$A = 5$
$\omega = 2$
$\phi = 0$



$A = 10$
$\omega = 3$
$\phi = 0$





A = 10
ω = 2
ϕ = 2π/7

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In the sine wave equation

$$\mathbf{x} = \mathbf{A}\sin(\omega t + \phi)$$

The value of ω is

$$\omega = \sqrt{\frac{k}{m}}$$

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Now we know what this function looks like. Let's check out the derivative and second derivative:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\cos(\omega t + \phi)\omega$$

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{x}}{dt}\right) = -\mathbf{A}\sin(\omega t + \phi)\omega^2 = -\omega^2\mathbf{x}$$

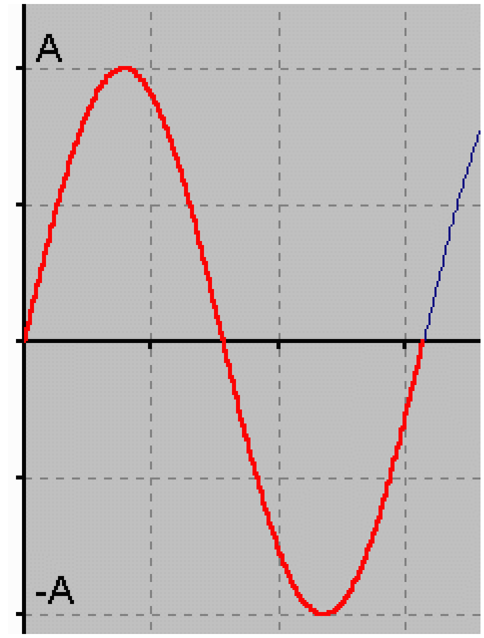
Substituting **x** and $d^2\mathbf{x}/dt^2$ into the differential equation yields

$$-\omega^2\mathbf{x} + \frac{k}{m}\mathbf{x} = 0$$

which means the equation is solved if

$$\omega = \sqrt{\frac{k}{m}}$$

ω is a mathematical quantity known as angular frequency. To figure out what ω is in terms of the measurable values such as Period (T) and Frequency (f), we need to recall that the period is the time required for the oscillator to go through one complete cycle. If, as shown in the picture, the oscillator starts at x = 0, it will be through one full cycle when the argument of the sine function (ωt) is equal to 2π. At one full cycle t = T, so

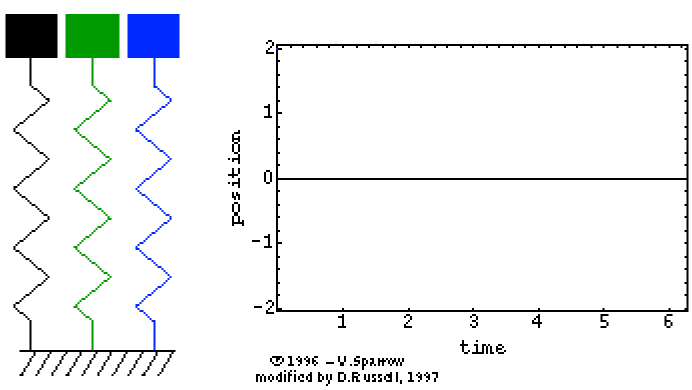


$$\omega T = 2\pi$$

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Note that the Amplitude of the SHO has no impact on its period (or frequency). The only factors that influence period are mass and spring constant.

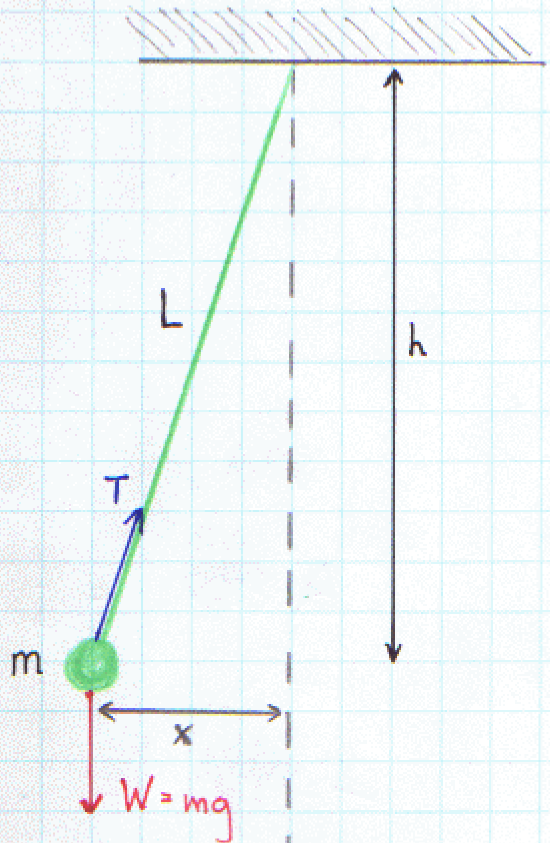
Every SHO contains a "mass" or "inertia" factor represented by the m in our equation for ω. Every SHO contains a "springiness" factor represented by k in our equation. In a spring-mass oscillator m and k are obvious. In other kinds of SHO's they may not be so obvious.



http://www.kettering.edu/~drussell/Demos/SHO/mass.html

Example Problem #1

A pendulum of length L with bob of mass m is a SHO if its swing has a small amplitude. Derive an equation for the period of a pendulum.



Only two forces act on the pendulum - gravity and the tension in the string.

These are shown in the diagram.

The tension force can be divided into x + y components.

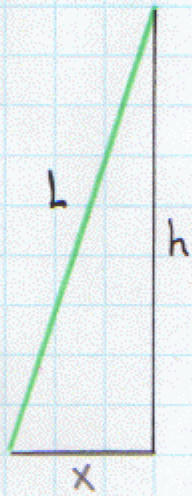
$$\Sigma F_x = T_x$$

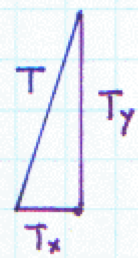
$$\Sigma F_y = T_y - W$$

If there is negligible vertical motion then $\Sigma F_y = 0$. This would only occur if the pendulum had a low-amplitude swing.

$T_y = W$ (small amplitude swing)

$$T_y = mg$$





Now lets look at T_x:

These are similar triangles, so

$$\frac{h}{x} = \frac{T_y}{T_x}$$

$$T_x = \frac{x\,T_y}{h}$$

However $T_y = mg$ and if the pendulum has a small swing with little vertical motion $h = L$ (approx).

$$T_x = \frac{x\,(mg)}{L}$$

$$\vec{T}_x = -\left(\frac{mg}{L}\right)\vec{x}$$

$$\vec{F} = -k\ \vec{x}$$

Here we have a Hooke's Law force. The force always acts opposite the displacement $\vec{x}$, hence the negative sign.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{mg}{L}}{m}}$$

$$\omega = \sqrt{\frac{g}{L}} \quad \text{for a pendulum.}$$

The period can now be found using $T = 2\pi/\omega$:

$$T = 2\pi\sqrt{\frac{L}{g}}$$ for a pendulum.

For a period of 1 second:

$1\ s = 2(3.1416)[L/g]^{0.5}$

Square both sides…

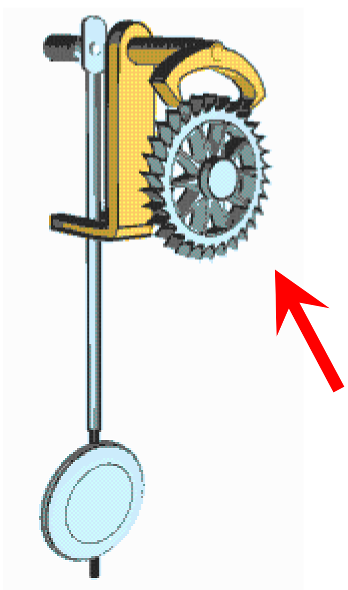
$1\ s^2 = 39.4784\ L/(9.8\ m/s^2)$

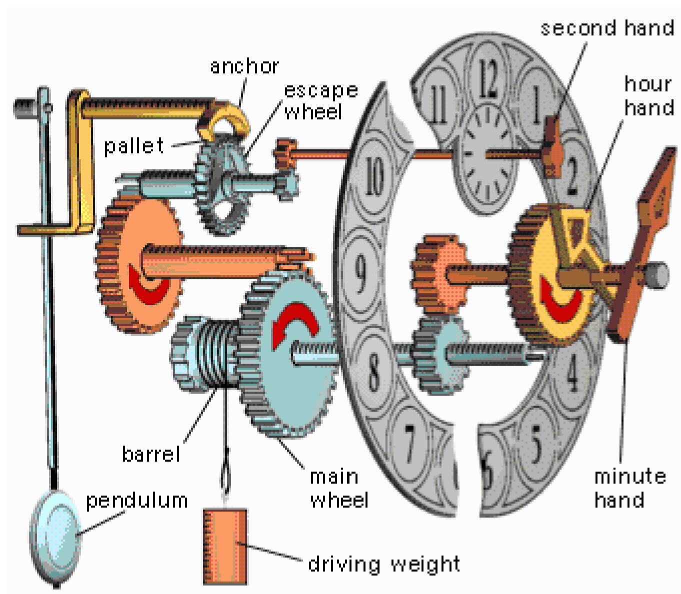
$(9.8\ m/s^2)\ 1\ s^2 = 39.4784\ L$

$L = 0.248\ m$

For a 1 second period the pendulum would need to be 24.8 cm long.

This formed the basis of a major advancement in technology – the development of the pendulum clock. Check out the animation below.





http://www.britannica.com/clockworks/pendulum.html