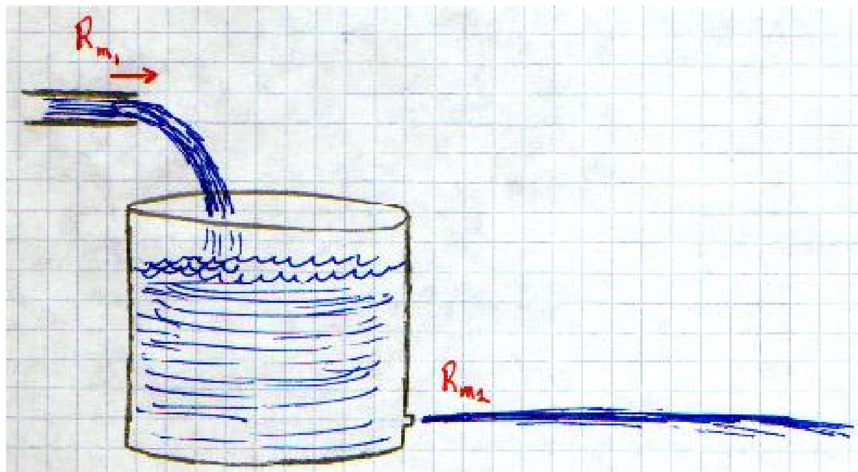
DAY 23

Summary of Primary Topics Covered

The Equation of Continuity

Consider the situation where fluid is flowing through some system and is in a state of dynamic equilibrium (meaning that, while the fluid is in motion, nothing in the system as a whole is changing). In the figure at right, water is added to a tank that has a leak in it.



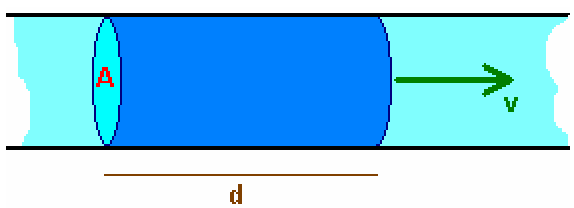
The height of water in the tank will reach a level at which the rate the water leaks out equals the rate water is coming in. At that point the height becomes steady and there are no further changes in the system. It is in *dynamic equilibrium*.

In dynamic equilibrium the rate of flow of fluid into the system (in terms of mass) equals the rate of flow of fluid out of the system. $R_{m1} = R_{m2} = R_m$

This means that is a given time, mass in equals mass out. $m_1/t = m_2/t = R_m$

Mass is density (ρ) times volume (Vol), $\rho_1 Vol_1/t = \rho_2 Vol_2/t = R_m$

and the volume is the cross-sectional area of the fluid times distance (d). $\rho_1 A_1 d_1/t = \rho_2 A_2 d_2/t = R_m$



Distance divided by time is speed. This is mass flow rate (measured in kg/sec) equation or the *equation of continuity*. $\rho_1 A_1 v_1 = \rho_2 A_2 v_2 = R_m$

If the fluid is incompressible, then the density does not change and ρ_1 equals ρ_2. If we divide through by ρ then density factors out of the equation and we get a continuity equation that depends on the volume flow rate (measured in m^3/sec, gal/sec, etc.).

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 = R_m$$

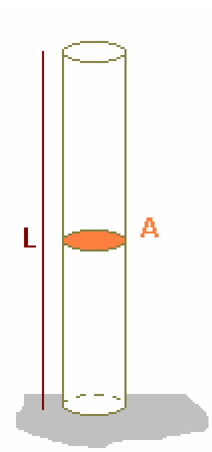
$$A_1 v_1 = A_2 v_2 = R_m/\rho = m/(\rho t)$$

$$A_1 v_1 = A_2 v_2 = Vol/t = R_{Vol}$$

$$A_1 v_1 = A_2 v_2 = R_{Vol}$$

Note that sometimes we use F to represent the flow rate, too. We ought to use R, because F is already used for force, but it's hard to not think of flow as F, too.

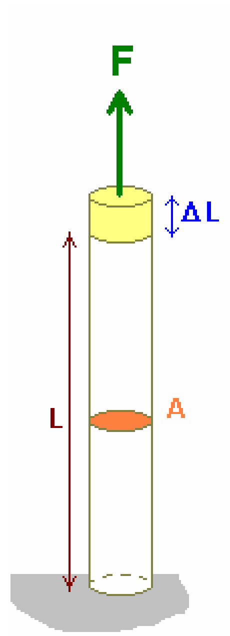
Elasticity



A piece of steel is imbedded in concrete. It has length L and cross-sectional area A. If a tension force is applied to the steel, the steel deforms (stretches) some (ΔL).

How much the steel stretches will depend on the area. A fatter piece of steel will stretch less than a thinner one. This is because there is less stress in a fat piece of steel because

Stress = F/A



Likewise, a long piece of steel will stretch more than a short one. Stretching a 4-inch long bolt by 1 inch is a lot of deformation. Stretching a 500 yard cable of the same diameter by 1 inch is not so significant. The amount of deformation is called strain

Strain = $\Delta L/L$

The ratio between stress and strain in an elastic material is linear and is basically a form of Hooke's Law:

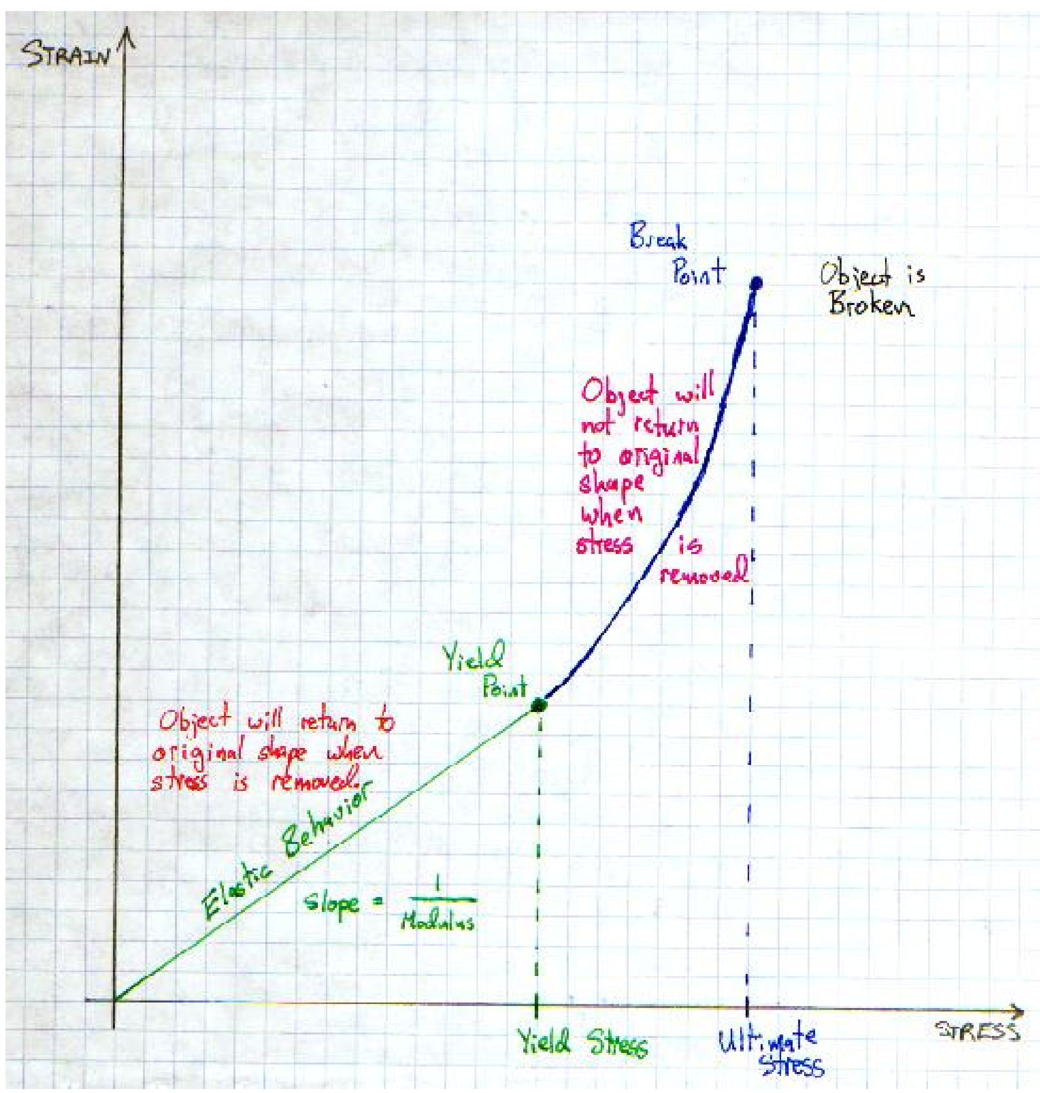
Stress = Modulus x Strain

Stress is that which causes deformation (akin to the force in Hooke's Law). Strain is a measure of amount of deformation (akin to x in Hooke's Law). Modulus relates the two -- akin to spring constant in Hooke's Law. In fact, the concepts of Modulus and Spring Constant are closely related, except that spring constant applies to a particular object (i.e. a *steel* spring 10 cm long, 2 cm wide, and 2 mm thick), whereas Modulus applies to a material (i.e. *steel* itself).

A material such as steel has several different moduli – one for a tension/compression forces (usually called Young's Modulus), one for shearing forces, one for bulk compression by immersion in a fluid, etc. One can usually look up these properties in a table.

Beyond Elasticity – Yielding and Breaking

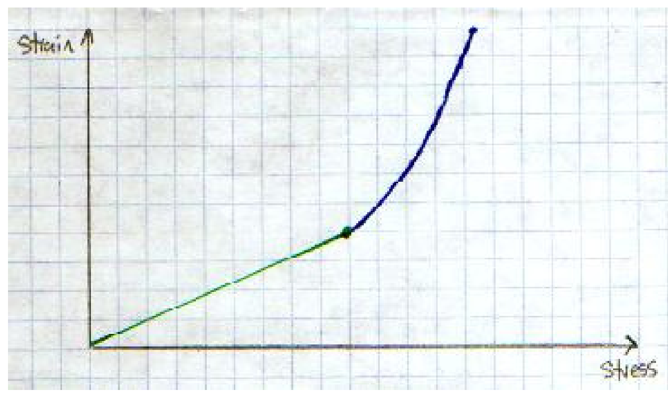
Materials only show elastic behavior within limits – those limits being that the strain the object is subjected to is small enough that the object will return to its original shape when the deforming stress is released. Deform any object too much and it will no longer return to its original shape – it has been deformed past its yield point. Keep deforming an object past the yield point and it tends to

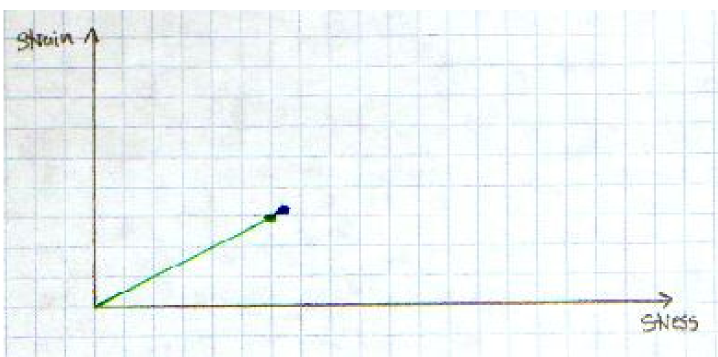


deform more dramatically. Eventually it will break. A plot of the deformation of typical object (its strain) vs. the deforming stress is shown on the previous page. When the object is within its elastic limit, the strain-stress plot is linear. Strain is proportional to stress. But once the stress exceeds the elastic limit and the object yields the plot gets more complicated.

The stress required to deform an object to the point of yielding is called the *Yield Stress* or *Yield Strength*. The stress required to break an object is called the *Ultimate Stress* or *Ultimate Strength*. Again, these are properties of a material and can usually be found in tables.

Materials such as copper will deform greatly once pushed past their elastic limit. Their yield point and break point are widely separated. These kinds of materials are said to be *ductile*. A plot of strain vs. stress for a ductile material is shown at right.





Materials such as glass will deform elastically, but once pushed past their yield points they break almost immediately. Their yield point and break point are nearly identical, as seen in the plot at left.

Example Problem #1:

At what speed must water flow out of a ¼ inch diameter hole if it is to fill a 5-gallon bucket in 4 minutes?

$r = .25 \text{ inch}/2 = .125 \text{ inch} = 3.175 \times 10^{-3} \text{ m}$
$A = \pi r^2 = 3.16693 \times 10^{-5} \text{ m}^2$

$R_{Vol} = 5 \text{ gal}/4 \text{ min} = 5 \text{ gal}/240 \text{ sec} = 5(3.786 \times 10^{-3} \text{ m}^3)/240 \text{ sec} = 7.8875 \times 10^{-5} \text{ m}^3/\text{s}$

$R_{Vol} = A\ v$

$7.8875 \times 10^{-5}\ m^3/s = 3.16693 \times 10^{-5}\ m^2\ (V) =$

$V = 2.49\ m/s$

Example Problem #2:

What tension is required to get a half-inch diameter, 4-inch long steel bolt to yield? How much will it have stretched at this point? What tension will break the bolt? How much will it have stretched at this point? Give values in English units. Use the short table of elastic properties. These values are approximate.

Young's Modulus for steel is 200 GPa ($200 \times 10^9\ N/m^2$).
The yield stress is 250 MPa ($250 \times 10^6\ N/m^2$).
The ultimate stress is 400 MPa ($250 \times 10^6\ N/m^2$).

Diam = .5 inch
L = 4 inch
r = .25 inch (1 m/39.37 inch) = .00635 m
$A = \pi r^2 = 1.26677 \times 10^{-4}\ m^2$

Yield Stress = F/A
$250 \times 10^6\ N/m^2 = F/1.26677 \times 10^{-4}\ m^2$
F = 31669.34412 N
F = 7000 lb to yield

Stress = Modulus x Strain
$250 \times 10^6\ N/m^2 = 200 \times 10^9\ N/m^2 \times$ Strain
.00125 = Strain

$\Delta L/L$ = Strain
$\Delta L/4$ inch = .00125
ΔL = 0.005 inches of stretch at yield point

Ultimate Stress = F/A
$400 \times 10^6\ N/m^2 = F/1.26677 \times 10^{-4}\ m^2$
F = 50670.95059 N
F = 11,000 lb to break

We can't determine how much the bolt will have stretched at the break point. The stress/strain relationship is not linear after the yield point.
Stress = Modulus x Strain
only works inside the elastic limit. However, we can say for sure that the bolt will have stretched more than 0.005 inches.

Example Problem #3:

A 1 inch diameter hose has a nozzle at the end that has a ¼ inch diameter hole in it. Water flows through the hose at a rate such that it takes 15 seconds to fill a gallon bucket. The nozzle sits on the ground and is aimed upward at a 45° angle.

Find the following:

- ❑ The speed of the water through the hose.
- ❑ The speed at which the water emerges from the nozzle.
- ❑ The maximum height the stream of water reaches after leaving the nozzle.
- ❑ The distance from the nozzle that the stream hits the ground.
- ❑ The thickness of the stream when it is at maximum height.
- ❑ The pressure in the hose.

$\rho_{water} = 1000\ kg/m^3$

Volume Flow Rate

$R_{V_1} = \frac{1\ gal}{15\ sec}$

$= \frac{3.785 \times 10^{-3}\ m^3}{15\ sec}$

$R_{V_1} = 2.523 \times 10^{-4}\ m^3/sec$

Diameters and Areas

$d_{hose} = 1"\left(\frac{1m}{39.37"}\right) = .0254\ m$

$A_{hose} = \pi r^2 = \pi\left(\frac{.0254m}{2}\right)^2 = 5.0671 \times 10^{-4}\ m^2$

$d_{nozzle} = \frac{1}{4}" = .00635\ m$

$A_{nozzle} = 3.167 \times 10^{-5}\ m^2$

I'll use continuity equation for incompressible fluids to find speeds:

$R_{vol} = Av$

$v = \frac{R_{vol}}{A}$

$v_{hose} = \frac{2.523 \times 10^{-4}\ m^3/s}{5.0671 \times 10^{-4}\ m^2} = .498\ m/s$

$v_{nozzle} = \frac{2.523 \times 10^{-4}\ m^3/s}{3.167 \times 10^{-5}} = 7.968\ m/s$

Bernoulli's for Pressure:

No heigh difference so these cancel.

$P_{hose} + \rho g y_{hose} + \frac{1}{2}\rho v_{hose}^2 = P_{nozzle} + \rho g y_{nozzle} + \frac{1}{2}\rho v_{nozzle}^2$

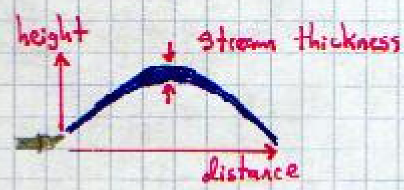
$P_{nozzle} = 1\ atm = 14.7\ Psi = 1.013 \times 10^5\ \frac{N}{m^2}$

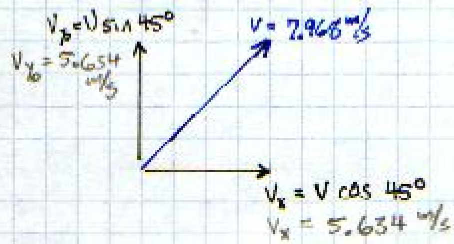
Because nozzle is exposed to air the water at the nozzle opening will be at atmosphere pressure

$P_{Hose} + \frac{1}{2}(1000\ \frac{kg}{m^3})(.498\ m/s)^2 = 1.013 \times 10^5\ \frac{N}{m^2} + \frac{1}{2}(1000\ kg)(7.968\ m/s)^2$

$P_{Hose} = 1.329 \times 10^5\ N/m^2$

$= 1.329 \times 10^5\ \frac{N}{m^2}\left(\frac{14.7\ psi}{1.013 \times 10^5\ \frac{N}{m^2}}\right) = 19.3\ psi$ or 4.6 psi above atmospheric pressure.





Find Height using vertical motion:

$V_{yo} = 5.634$ m/s

$V_y = 0$ (at top of arc)

$a_y = -9.8$ m/s²

$y_o = 0$

$y = ?$

$V^2 = V_o^2 + 2a(y - y_o)$

$0^2 = (5.634 \text{ m/s})^2 + 2(-9.8 \text{ m/s}^2)(y - 0)$

$y = \dfrac{(5.634 \text{ m/s})^2}{19.6 \text{ m/s}^2} = 1.620\text{m}$ or **5.31 ft**

Find distance using vertical + horizontal motion:

First vertical to find time in air:

$y_o = 0$ & $y = 0$ (for full arc)

$a_y = -9.8$ m/s²

$V_{yo} = 5.634$ m/s

$y = y_o + V_{oy}t + \frac{1}{2}at^2$

$0 = 0 + 5.634 \text{ m/s}\, t - 4.9 \text{ m/s}^2\, t^2$

$4.9 \text{ m/s}^2\, t = 5.634$ m/s

$t = 1.150$ sec

Now horizontal to find distance.

$x = V_x t = 5.634 \text{ m/s}(1.150\text{ s})$

$x = 6.478$ m = **21.3 ft**

At top of arc the stream only has horizontal velocity so $V = 5.634$ m/s.
Use continuity equation now to find area of the stream's cross-section.

$R_{vol} = Av$

$A = \dfrac{R_{vol}}{v} = \dfrac{2.523 \times 10^{-4} \text{ m}^3/\text{s}}{5.634 \text{ m/s}} = 4.479 \times 10^{-5} \text{ m}^2 = \pi r^2$

$r = .003776$ m $= .149$ in

$d = 2r = .297$ inch

Answers:

The speed of the water through the hose:

0.5 m/s or 1.6 ft/sec

The speed at which the water emerges from the nozzle:

8.0 m/s or 26.1 ft/sec

The maximum height the stream of water reaches after leaving the nozzle:
5.3 ft
The distance from the nozzle that the stream hits the ground:
21.3 ft
The thickness of the stream when it is at maximum height:
.297 inch
The pressure in the hose:
19.3 psi or 4.6 psi above atmospheric pressure

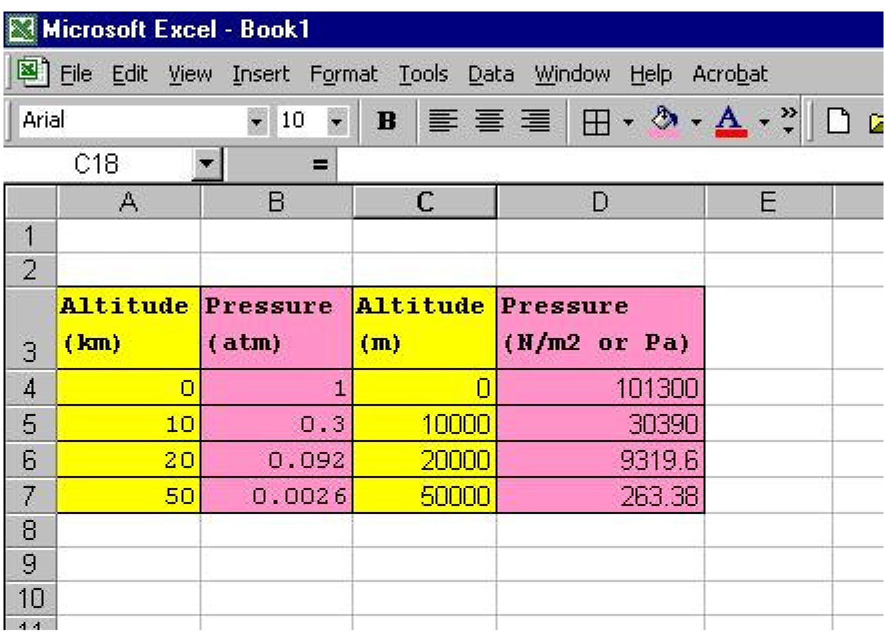
PHY 231 ONLY
Example Problem #4:

Below is data giving the pressure in the troposphere for a variety of altitudes:

Altitude (km)	**Pressure (atm)**
0	1.0
10	0.30
20	0.092
50	0.0026

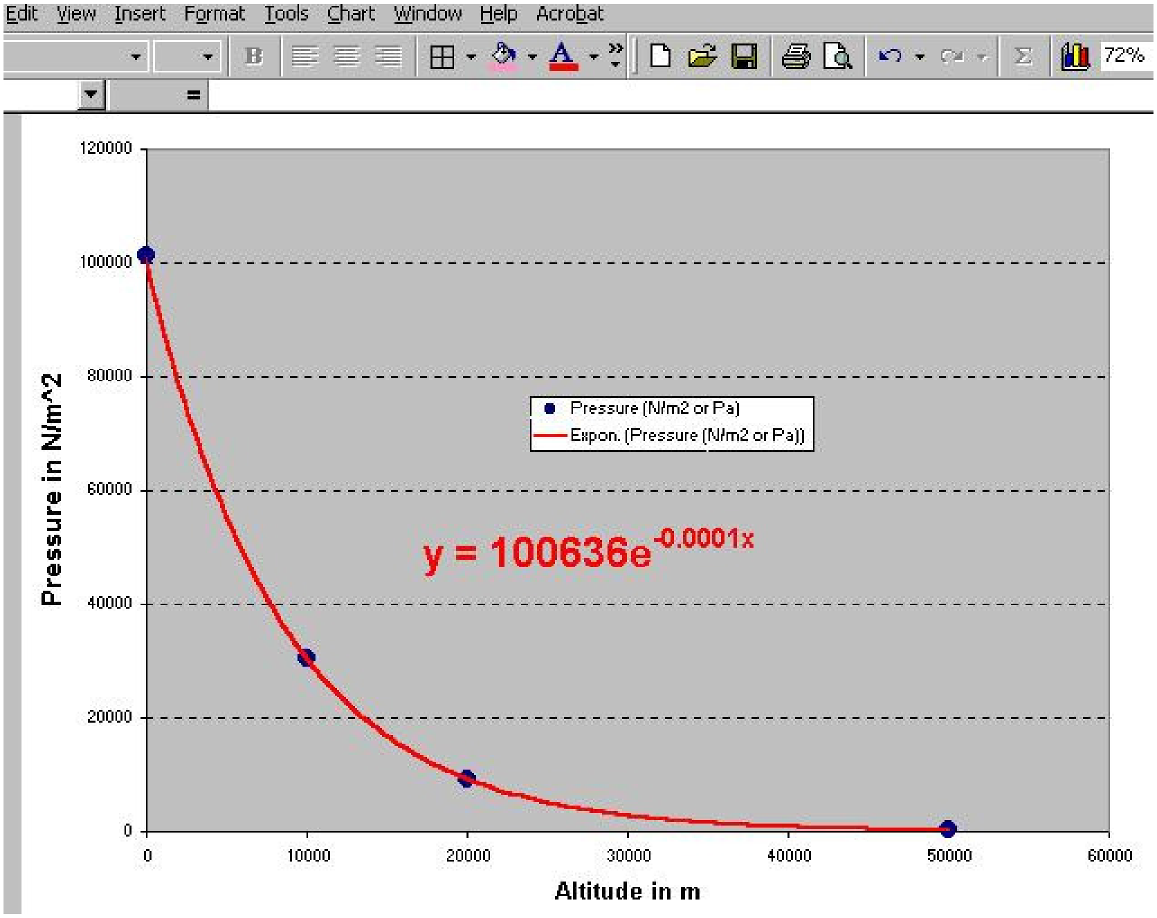
This is due to fluid pressure in a compressible fluid (air). Pressure in the lower atmosphere generally falls off exponentially as you go up. Use EXCEL to obtain an equation for Pressure in Pa as a function of altitude in m, then use that equation to calculate the density of air at sea level, at an altitude of 1 mile (the height of Denver), and at 29,035 feet (the height of Mt. Everest).

First I enter the data into EXCEL and use EXCEL to convert to SI units:



	A	B	C	D	E
1					
2					
3	Altitude (km)	Pressure (atm)	Altitude (m)	Pressure (N/m2 or Pa)	
4	0	1	0	101300	
5	10	0.3	10000	30390	
6	20	0.092	20000	9319.6	
7	50	0.0026	50000	263.38	
8					
9					
10					

Then I make a graph and fit an exponential to the data:



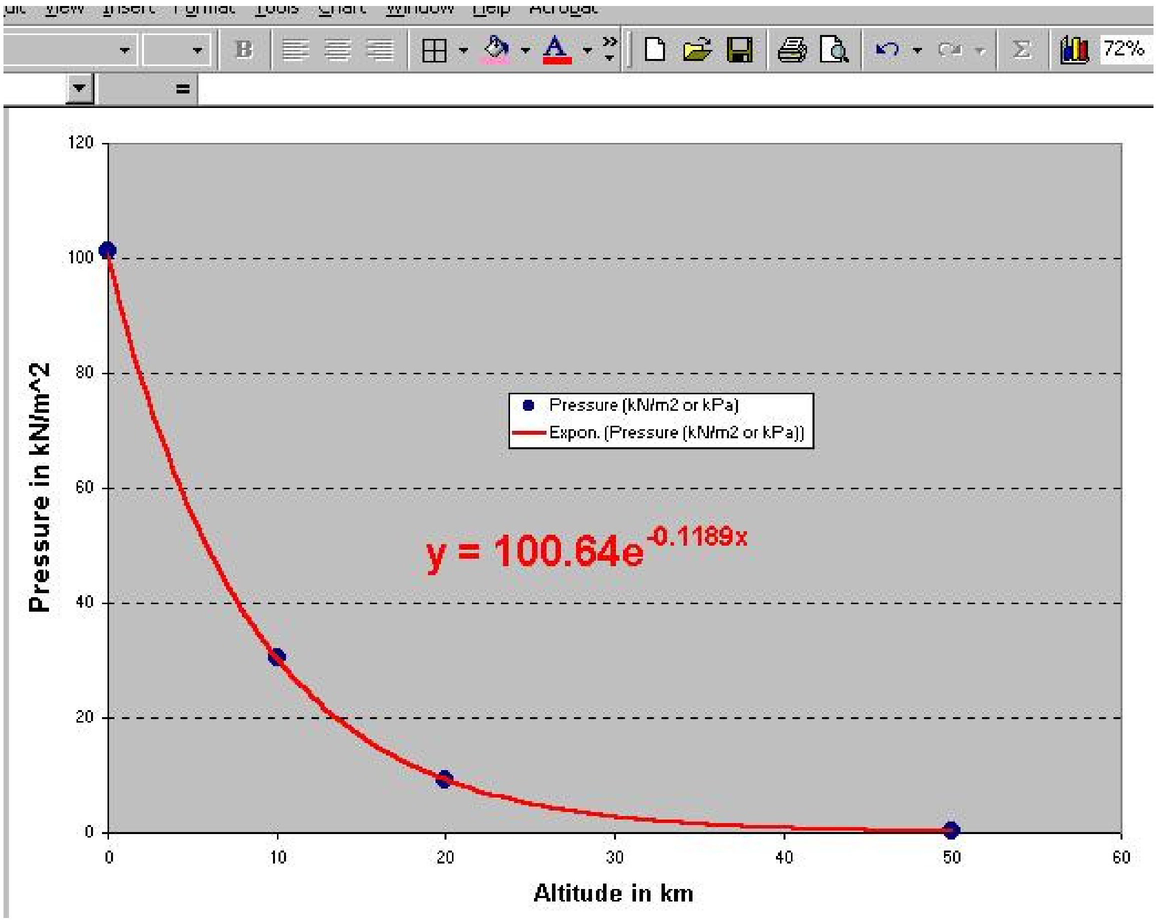
So my equation for Pressure is

$P = 100636\ e^{-0.0001y}$

Where y is altitude above the ground. I don't trust that 0.0001 – I bet there's a big rounding error there. Let me re-plot, using km and kPa. Here's the figures:

Altitude (km)	Pressure (atm)	Altitude (km)	Pressure (kN/m2 or
0	1	0	101.3
10	0.3	10	30.39
20	0.092	20	9.3196
50	0.0026	50	0.26338

And here's the graph:



Plotting with different units helped. It looks like my equation should be

$P = 100636\ e^{-0.0001189y}$

Of course I could also have just changed the number of decimal points that EXCEL displayed but I wanted to do it this way.

Now I know that for fluid pressure, $dP = \rho\ g\ dy$. So

$dP/dy = \rho\ g$

The derivative of Pressure with respect to altitude is density times g. So density is

$\rho = (dP/dy)/g$

Now I work it out:

$$dP/dy = -0.0001189\,(100636)\,e^{-0.0001189y}$$ Units are Pa/m
$$= -11.966\,e^{-0.0001189y}$$

$$\rho = (-11.966\,e^{-0.0001189y})/g$$ Units are Pa/m per m/s^2.
$$= (-11.966\,e^{-0.0001189y})/(-9.8)$$ Let's work out those units…

$$\frac{\frac{Pa}{m}}{\frac{m}{s^2}} = \frac{\frac{N/m^2}{m}}{\frac{m}{s^2}} = \frac{N}{m^3} * \frac{s^2}{m} = \frac{Ns^2}{m^4} = \frac{(kgm/s^2)s^2}{m^4} = \frac{kg}{m^3}$$

…and the units are density units!

$\rho = 1.221\,e^{-0.0001189y}$ Units are kg/m^3

Now to find the densities at sea level, Denver, and Everest:

Sea Level **y = 0:**

$\rho = 1.221\,e^{-0.0001189(0)}$ = 1.22 kg/m^3

(that agrees pretty well with the 1.29 kg/m^3 value we usually use).

Denver **y = 1 mile = 1609 m:**

$\rho = 1.221\,e^{-0.0001189(1609)}$ = 1.00 kg/m^3

Everest **y = 29035 ft = 8849.44 m:**

$\rho = 1.221\,e^{-0.0001189(8849.44)}$ = .46 kg/m^3

(the air is pretty thin up there).