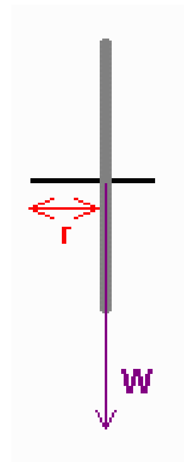
DAY 20 -- PHY 231 ONLY

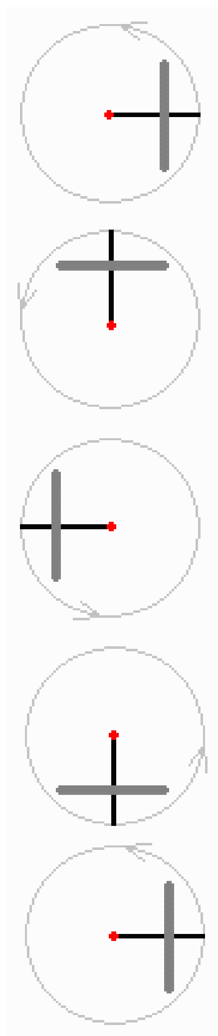
Summary of Primary Topics Covered

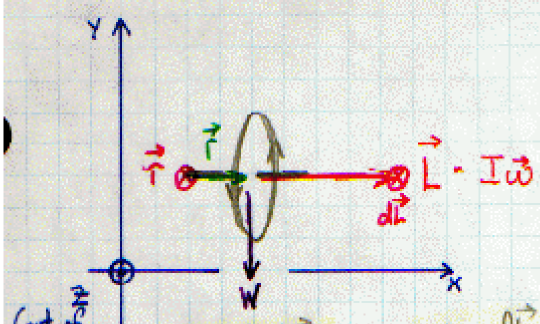
Using Calculus with Vectors

Using calculus we can solve vector problems that we can't solve with algebra, and we can learn some more about problems that we can solve with algebra.

We can calculate the rate of precession of a gyroscope. At right is an edge-on view from the side, showing that gyroscope has a weight of W, and sits at the end of an arm of length r. Below is a figure showing the gyroscope precessing through 360°, as seen from above. The solution is worked out below.







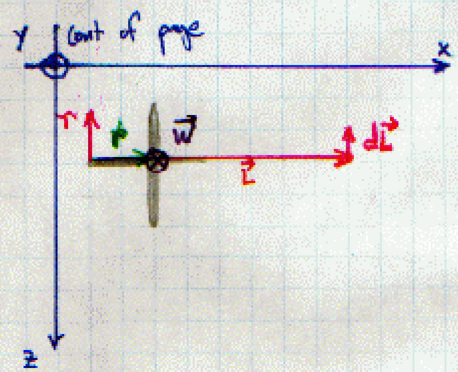
$\vec{\tau} = \vec{r} \times \vec{W}$ $\qquad$ $\vec{\tau} = \frac{d\vec{L}}{dt}$

Since dt is a scalar
$d\vec{L}$ points in same direction as $\vec{\tau}$

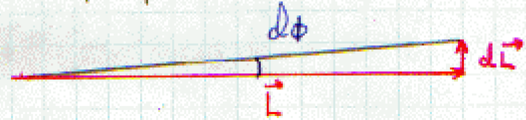
$\tau = rW$ directed into page (along neg z axis)

$$rW = \frac{dL}{dt}$$

Viewed from above (looking down y axis)

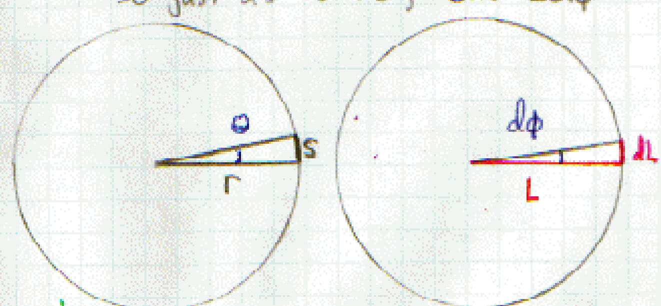


During tiny time dt the gyroscope precesses by amount $d\phi$.



Since we are dealing with differentials (infinitesimals) dL cay be thought of as a tiny segment of a circle with radius L

So just as $s = r\theta$, $dL = L\,d\phi$



$$rW = \frac{dL}{dt} = \frac{L\,d\phi}{dt}$$

$$\frac{rW}{L} = \frac{d\phi}{dt}$$

$$\frac{rW}{I\omega} = \frac{d\phi}{dt}$$

$d\phi$ is the amount the gyroscope precesses in time dt.
So $\frac{d\phi}{dt}$ is the Rate of Precession of the gyroscope. We'll call this Ω

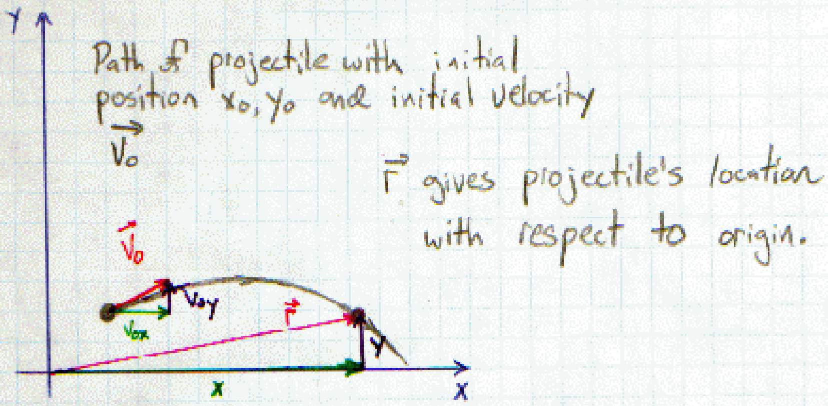
$$dL = L\,d\phi$$

$$\Omega = \frac{rW}{I\omega}$$

With the equation **Ω = rW/Iω**, we see that the rate of precession is greater if the gyroscope wheel is heavier (W) or if the arm the gyroscope is precessing around is longer (r). The rate of precession is lower if the wheel is spinning more rapidly (ω) or if it has a higher moment of inertia (I).

We can also apply calculus to determine velocity and acceleration for almost any vector problem. If we take a calculus and vectors approach to projectile motion, we don’t get any different results than we got with the algebra approach that we used for projectile motion before, but we do get to look at the problem in a different way.

Perhaps in the case of projectile motion there is no obvious advantage to the use of calculus over algebra. However, for other kinds of motion using the exact same approach with calculus can tell us things that algebra cannot tell us. In example problem #2 use of calculus shows us clearly that the acceleration of an object moving in a circular path is directed toward the center of that circle – and hence is referred to as *centripetal* (or center-seeking) acceleration.



$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$$

Horizontal Motion: Starting position x_0; Starting velocity V_{0x}; Accel. $a_x = 0$

$$X = x_0 + V_{0x}t + \tfrac{1}{2}\cancelto{0}{a_x}t^2$$

Vertical Motion: Starting position y_0; Starting velocity V_{0y}; Accel. $a_y = -9.8\ \text{m/s}^2$

$$y = y_0 + V_{0y}t - 4.9t^2$$

$$\vec{r} = (x_0 + V_{0x}t)\,\hat{\imath} + (y_0 + V_{0y}t - 4.9t^2)\,\hat{\jmath}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x_0 + V_{0x}t)\,\hat{\imath} + \frac{d}{dt}(y_0 + V_{0y}t - 4.9t^2)\,\hat{\jmath}$$

$$\vec{v} = V_{0x}\,\hat{\imath} + (V_{0y} - 9.8t)\,\hat{\jmath}$$

y direction velocity changes with time

x-direction velocity is constant and equal to

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(V_{0x})\,\hat{\imath} + \frac{d}{dt}(V_{0y} - 9.8t)\,\hat{\jmath}$$

$$= 0\,\hat{\imath} + (-9.8)\,\hat{\jmath}$$

$$\vec{a} = -9.8\,\hat{\jmath}$$

Acceleration is in negative $\hat{\jmath}$ (y) direction only and equal to a constant 9.8 m/s²

PHY 231 ONLY Example Problem #1:

Show that for a gyroscope that uses a disk as a wheel, the mass of the gyroscope has no effect on the precession rate.

If the gyroscope uses a disk as a wheel, then $I = \frac{1}{2}mR^2$, where R is the radius of the wheel. The weight is $W = mg$.

$\Omega = rW/I\omega$

$\Omega = r(mg)/(\frac{1}{2}mR^2)\omega$ substitute for W & I

$= rg/(\frac{1}{2}R^2)\omega$ masses cancel out

$\Omega = 2rg/\omega R^2$

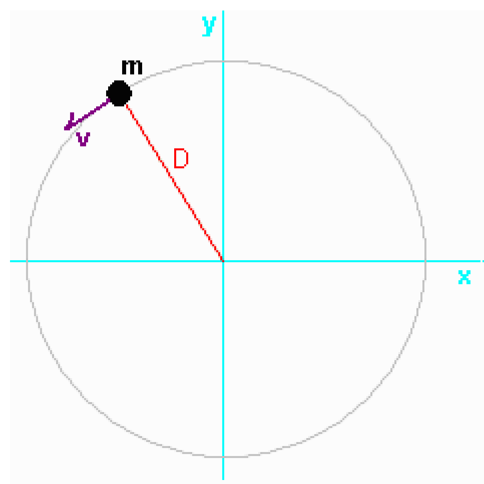
So Ω does not depend on m at all. The precession rate depends only on the dimensions of the gyroscope (r & R), the gravitational field strength (g), and the rate at which the wheel is spinning (ω).

PHY 231 ONLY Example Problem #2:

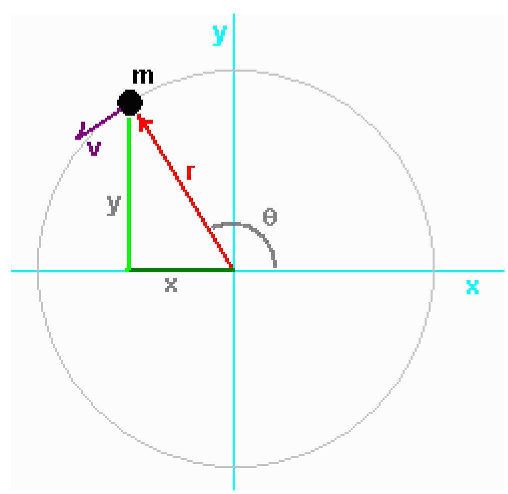
A mass m is moving in a horizontal circle on the end of a string of length D. The mass is moving with tangential velocity v.

Find an equation for the position vector of the mass. Then use calculus to find the acceleration of the mass and the force acting on the mass in terms of m, D, and v.

Determine the force on the mass for the following data: m = 2 kg, D = .75 m, v = 10 m/s.



The position vector of the mass is $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. x & y are the coordinates of the mass. The only thing that is changing in this problem is the angle that the string is making with the coordinate system, so let's illustrate that angle, and show $\mathbf{r}$, x & y:



$x = D\cos(\theta)$

$y = D\sin(\theta)$

also, θ is changing because m is moving.

$\theta = \omega t$, and $\omega = v/D$.

Note that the magnitude of the vector $\mathbf{r}$ is D, but the direction of $\mathbf{r}$ is constantly changing as the mass orbits the center of the circle. $\mathbf{r}$ always point <u>away</u> from the circle's center.

So now

$\mathbf{r} = D\cos(\theta)\ \mathbf{i} + D\sin(\theta)\ \mathbf{j}$

$\mathbf{r} = D\cos(\omega t)\ \mathbf{i} + D\sin(\omega t)\ \mathbf{j}$

Now to find acceleration, we take derivatives:

$$\mathbf{v} = d\mathbf{r}/dt = \frac{d}{dt}\,[D\cos(\omega t)\ \mathbf{i} + D\sin(\omega t)\ \mathbf{j}]$$

$$= D\,\frac{d}{dt}\,[\cos(\omega t)\ \mathbf{i} + \sin(\omega t)\ \mathbf{j}]$$

factor out D because the string length is constant

$$\mathbf{v} = D\,[-\sin(\omega t)\,\omega\ \mathbf{i} + \cos(\omega t)\,\omega\ \mathbf{j}]$$

$$\mathbf{a} = d\mathbf{v}/dt = \frac{d}{dt}\,\{D\,[-\sin(\omega t)\,\omega\ \mathbf{i} + \cos(\omega t)\,\omega\ \mathbf{j}]\}$$

$$= D\,\frac{d}{dt}\,[-\sin(\omega t)\,\omega\ \mathbf{i} + \cos(\omega t)\,\omega\ \mathbf{j}]$$

$$= D\,[-\cos(\omega t)\,\omega^2\ \mathbf{i} - \sin(\omega t)\,\omega^2\ \mathbf{j}]$$

$$\mathbf{a} = -\ \omega^2 D\,[\cos(\omega t)\ \mathbf{i} + \sin(\omega t)\ \mathbf{j}]$$

factor out the negative and ω^2

But if I multiply D through the quantity in brackets I end up with the equation for **r** that is hilited above:

$$\mathbf{a} = -\ \omega^2[D\cos(\omega t)\ \mathbf{i} + D\sin(\omega t)\ \mathbf{j}]$$

$$\mathbf{a} = -\ \omega^2[\mathbf{r}]$$

$$\mathbf{a} = -\ (v/D)^2\ \mathbf{r}$$

$\omega = v/D$

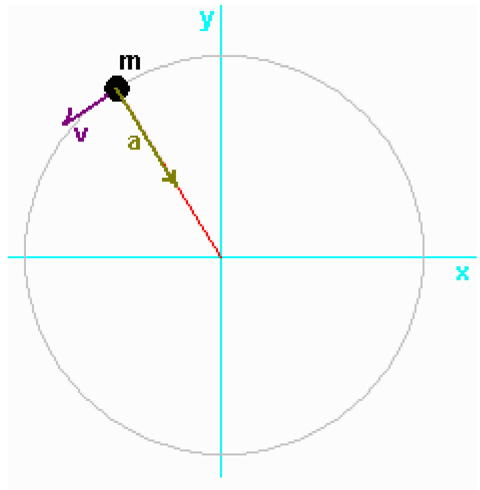
In terms of magnitudes, r = D so

$$a = (v/D)^2\ r$$

$$a = (v/D)^2\ D$$

$$a = v^2/D$$

In terms of direction, because of the minus sign, **a** points in the opposite direction as **r**. Since **r** always points away from the circle's center, **a** always points toward the circle's center.



So, the acceleration is $a = v^2/D$ toward the center of the circle. Because it is toward the center, it is a centripetal ("center-seeking") acceleration.

The force on the mass is

$$F = m\ a = m\ v^2/D$$

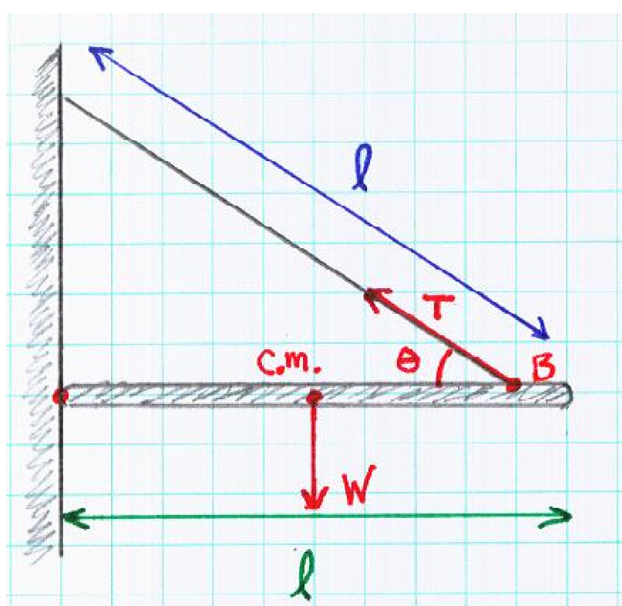
also toward the center of the circle. This is centripetal force. For the data given,

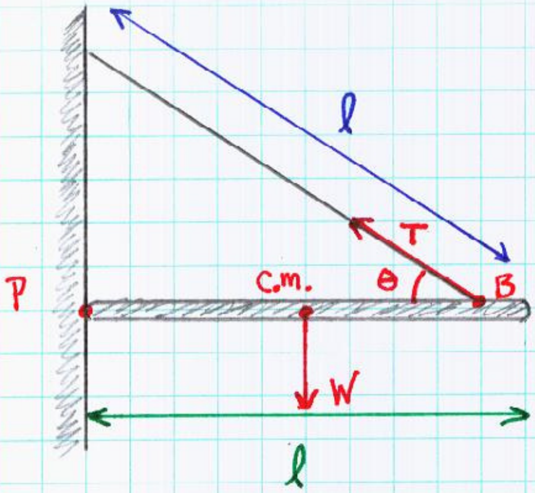
$$F = (2\ kg)(10\ m/s)^2/.75\ m = 267\ kgm^2/s^2/m = 267\ kgm/s^2 = 267\ N$$

PHY 231 ONLY Example Problem #3:

A beam of uniform density, mass m and length ℓ is attached to a vertical wall. The beam is horizontal. One end is attached to the wall by a hinge (which can pivot). The other end is supported by a cable, also of length ℓ, which runs from a point B on the beam up to the wall. Obtain an equation for the tension in the cable in terms of m, g, and the angle the cable makes with the bar (note that ℓ will not appear in the equation). Show that the tension in the cable is minimized if the angle it makes with the bar is 45°.

I start with making a diagram. The beam is uniform so its center of mass point (cm) is located in the middle of the beam.





$r_{P\text{-}CM}$ is distance from P to the CM point.

$r_{P\text{-}B}$ is distance from P to point B.

I'll work this the way I worked the sign example the other day.

It's a statics problem so the clockwise and counter-clockwise torques about P must be equal.

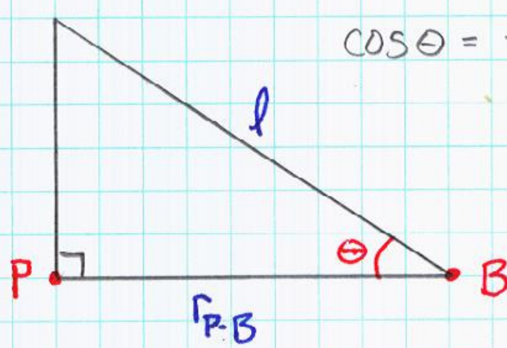
$$\tau_{CL} = \tau_{CCL}$$

$$r_{P\text{-}CM} W = r_{P\text{-}B} T \sin\theta$$

$$W = mg \quad \text{and} \quad r_{P\text{-}cm} = \tfrac{1}{2} l$$

$$\tfrac{1}{2} l (mg) = r_{P\text{-}B} T \sin\theta$$

$$T = \frac{lmg}{2 r_{P\text{-}B} \sin\theta}$$



$$\cos\theta = \frac{r_{P\text{-}B}}{l}$$

The triangle at right shows me that

$$r_{P\text{-}B} = l\cos\theta$$

$$T = \frac{l m g}{2 l \cos\theta \sin\theta}$$

$$T = \frac{mg}{2\cos\theta\sin\theta}$$

There's my equation and, sure enough, it doesn't have ℓ in it.

T is a minimum when $\frac{dT}{d\theta} = 0$

$$\frac{d}{d\theta}\left(\frac{mg}{2\cos\theta\sin\theta}\right) = 0 \rightarrow$$ Quotient Rule!

$$\frac{mg\frac{d}{d\theta}(2\cos\theta\sin\theta) - 2\cos\theta\sin\theta\left(\frac{d(mg)}{d\theta}\right)}{(2\cos\theta\sin\theta)^2} = 0$$

mg is constant so this is zero

$$\frac{2mg\frac{d}{d\theta}(\cos\theta\sin\theta) - 0}{(2\cos\theta\sin\theta)^2} = 0$$

Divide both sides by $2mg$. Multiply both sides by $(2\cos\theta\sin\theta)^2$. I'm left with

$$\frac{d}{d\theta}(\cos\theta\sin\theta) = 0 \rightarrow$$ Product Rule!

$$\cos\theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\cos\theta) = 0$$

$$\cos\theta\cos\theta + \sin\theta(-\sin\theta) = 0$$

$$\cos^2\theta = \sin^2\theta$$

$$1 = \frac{\sin^2\theta}{\cos^2\theta}$$

$$1 = \tan^2\theta$$

$$1 = \tan\theta$$

$$\tan^{-1}(1) = \theta = 45^\circ$$

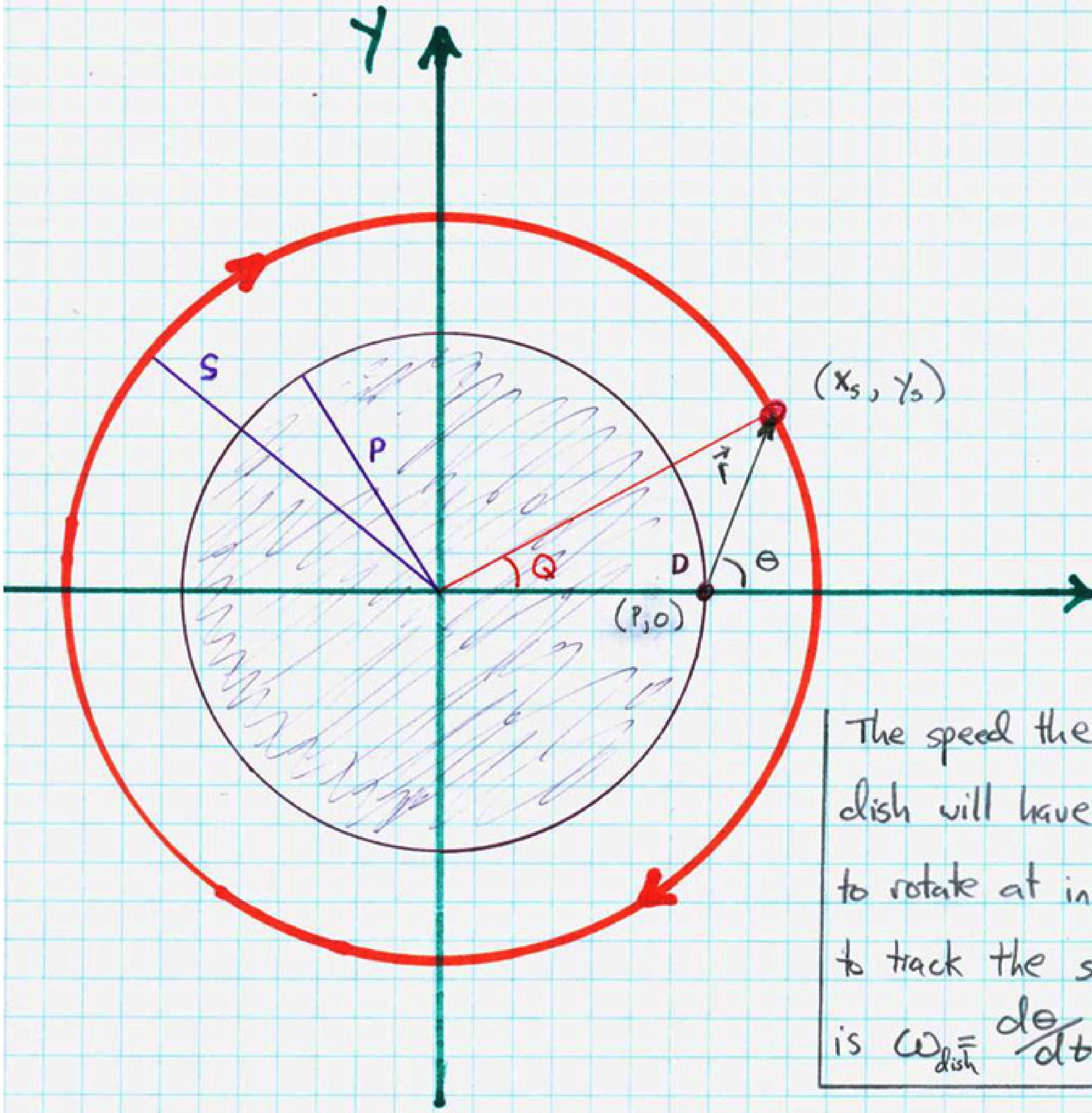
PHY 231 ONLY Example Problem #4:

A satellite orbits a planet. The planet has radius P and the satellite orbit has radius S. On the planet at point D is a tracking station with a dish antenna. As seen by an observer on the planet, the satellite has an angular velocity of w.

(a) Come up with an equation for the angular velocity the dish must have to track the satellite.

(b) If the planet is small relative to the satellite, the angular velocity of the dish should match that of the satellite. Show this both by graphing and by manipulating the equation you got in part (a).

I start with making a diagram showing the planet, the satellite's orbit, and the dish's location. The location of the satellite is at (x_S, y_S). I've oriented my drawing so the dish has location (P,0). Then I draw in the position vector of the satellite as seen from the dish.

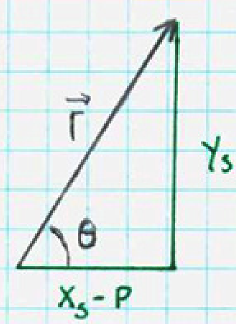


The speed the dish will have to rotate at in order to track the satellite is $\omega_{dish} = \frac{d\theta}{dt}$

Position of satellite with respect to dish:

$$\vec{r} = (x_s - P)\,\hat{\imath} + y_s\,\hat{\jmath}$$

$$\theta = \tan^{-1}\left(\frac{y_s}{x_s - P}\right)$$



$$X_S = S\cos Q \qquad Q = \omega t$$

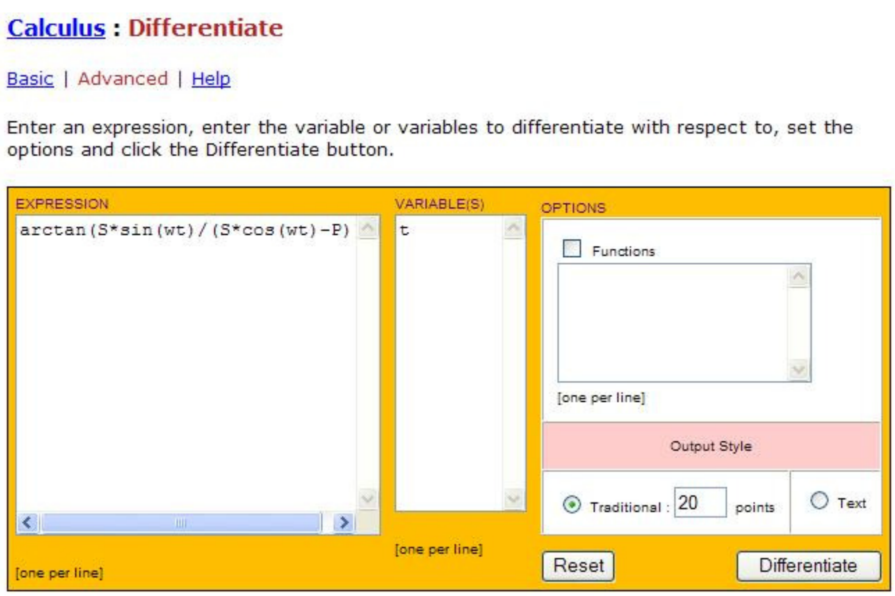
$$Y_S = S\sin Q$$

$$\theta = \tan^{-1}\left[\frac{S\sin(\omega t)}{S\cos(\omega t) - P}\right]$$

$$\omega_{Dish} = \frac{d\theta}{dt}$$

To do this derivative I use the QuickMath web page. Hey – that is one UGLY derivative! Who wants to do THAT by hand? Not me. You could use a good calculator, too.

Type it in (I had to look up the $\tan^{-1}$ function – the web page said I enter it as ARCTAN)…



…and press “Differentiate”….

Command

Differentiate

Expression

$$\tan^{-1}\left(\frac{S\sin(t\,w)}{S\cos(t\,w)-P}\right)$$

Variables

t

Result

$$\frac{\frac{S^2\,w\sin^2(t\,w)}{(S\cos(t\,w)-P)^2}+\frac{S\,w\cos(t\,w)}{S\cos(t\,w)-P}}{\frac{S^2\sin^2(t\,w)}{(S\cos(t\,w)-P)^2}+1}$$

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So here it is…

$$\omega_{Dish}=\frac{\frac{S^2\omega\sin^2(\omega t)}{(S\cos(\omega t)-P)^2}+\frac{S\omega\cos(\omega t)}{(S\cos(\omega t)-P)}}{\frac{S^2\sin^2(\omega t)}{(S\cos(\omega t)-P)^2}+1}$$

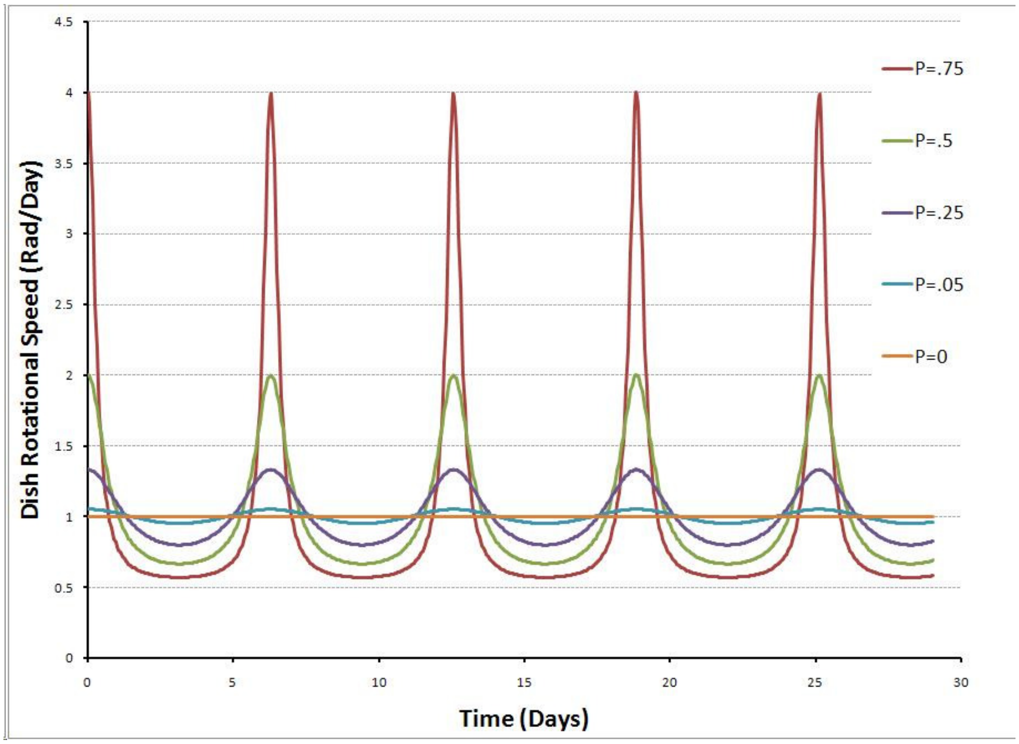
Now to show what happens when the planet is small…

First, I set up EXCEL or a calculator to graph the equation for the S=1, w=1, and P=.75 (this would mean the planet is 0.75 the radius of the satellite's orbit and the satellite moved at a rate of 1 rad/day)…

w | f_x 1

	A	B	C	D	E
1	i	0.05			
2	S	1			
3	P	0.75			
4	w	1			
5					
6	t	ω t (rad)	cos(ωt)	sin(ωt)	ω_{dish}
7	0	0	1	0	4
8	0.05	0.05	0.99875	0.049979	3.898079
9	0.1	0.1	0.995004	0.099833	3.625279
10	0.15	0.15	0.988771	0.149438	3.257004
11	0.2	0.2	0.980067	0.198669	2.867421
12	0.25	0.25	0.968912	0.247404	2.504465
13	0.3	0.3	0.955336	0.29552	2.189251
14	0.35	0.35	0.939373	0.342898	1.92563
15	0.4	0.4	0.921061	0.389418	1.709175
16	0.45	0.45	0.900447	0.434966	1.532671
17	0.5	0.5	0.877583	0.479426	1.388772
18	0.55	0.55	0.852525	0.522687	1.271025
19	0.6	0.6	0.825336	0.564642	1.174121
20	0.65	0.65	0.796084	0.605186	1.093825
21	0.7	0.7	0.764842	0.644218	1.026808

…and I do this for P=.5, P=.25, etc to see what happens for a smaller and smaller planet size – all the way down to P=0. I made graphs of all these (computers and graphing calculators are great).



I see that at P gets smaller the dish's motion comes closer to being just a constant 1 rad/day – the same as ω.

Now let's do this by just setting P=0 in my equation from (a)…

$$\omega_{Dish} = \frac{\dfrac{S^2\omega\sin^2(\omega t)}{(S\cos(\omega t)-P)^2} + \dfrac{S\omega\cos(\omega t)}{(S\cos(\omega t)-P)}}{\dfrac{S^2\sin^2(\omega t)}{(S\cos(\omega t)-P)^2} + 1}$$

If $P=0$

$$\omega_{Dish} = \frac{\dfrac{\cancel{S^2}\,\omega\sin^2(\omega t)}{\cancel{S^2}\cos^2(\omega t)} + \dfrac{\cancel{S}\,\omega\,\cancel{\cos}(\omega t)}{(\cancel{S}\,\cancel{\cos}(\omega t))}}{\dfrac{\cancel{S^2}\sin^2(\omega t)}{\cancel{S^2}\cos^2(\omega t)} + 1}$$

$$\omega_{Dish} = \frac{\dfrac{\omega \sin^2(\omega t)}{\cos^2(\omega t)} + \omega}{\dfrac{\sin^2(\omega t)}{\cos^2(\omega t)} + 1}$$

factor out ω

$$= \omega\left(\frac{\cancel{\dfrac{\sin^2(\omega t)}{\cos^2(\omega t)}} + 1}{\cancel{\dfrac{\sin^2(\omega t)}{\cos^2(\omega t)}} + 1}\right)$$

$$\omega_{Dish} = \omega$$