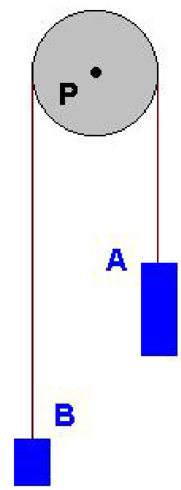
DAY 13

Summary of Primary Topics Covered

It may be easy to <u>say</u> "rotation is analogous to translation". Putting that knowledge to work will take some practice. That is what we did today – practiced working problems of real complexity.

Example Problem #1:



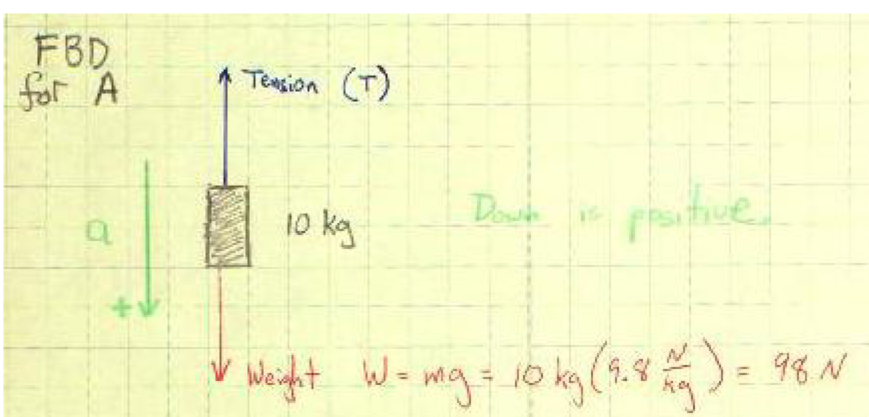
An "Atwood's Machine" consists of two masses connected by a light string that passes over a pulley.

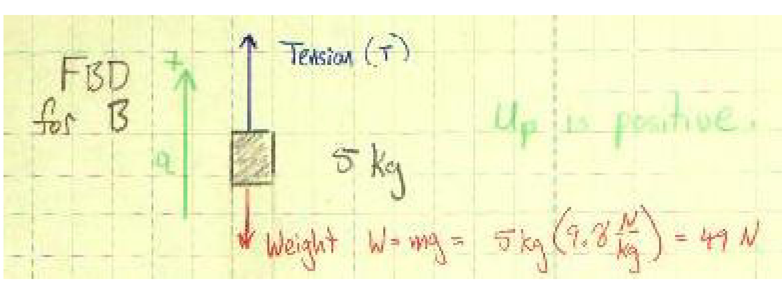
Mass A is 10 kg.
Mass B is 5 kg.

The pulley is massless and frictionless. Find the acceleration of the blocks.

This problem asks for acceleration so we know we will have to use forces and Newton's laws to solve it – not Energy or Momentum. Nothing that we've done with Energy or Momentum has involved acceleration.

First we draw Free Body Diagrams for both masses:





The tensions will be the same since the two blocks are essentially exerting forces on each other via the string. The accelerations will be the same because the two blocks are connected by the string. The only thing is that one will accelerate upward and the other will accelerate downward.

Then we apply Newton's 2nd Law of motion to both:

Newton's 2nd Law For A:

$$\sum F = ma$$

$$W - T = ma$$

$$\boxed{98\,N - T = (10\,kg)\,a}$$

A

Newton's 2nd Law For B:

$$\sum F = ma$$

$$T - W = ma$$

$$\boxed{T - 49\,N = (5\,kg)\,a}$$

B

That gives two equations in two unknowns. We can solve these two for "a":

$$98\,N - T = (10\,kg)\,a \qquad T - 49\,N = (5\,kg)\,a$$

substitute for T

$$T = (5\,kg)\,a + 49\,N$$

$$98\,N - \left((5\,kg)\,a + 49\,N\right) = (10\,kg)\,a$$

$$98\,N - 5\,kg(a) - 49\,N = 10\,kg(a)$$

$$49\,N = (15\,kg)\,a$$

$$\frac{49\ \frac{kg\,m}{s^2}}{15\,kg} = a$$

$$\boxed{3.27\ m/s^2 = a}$$

FINAL ANSWER

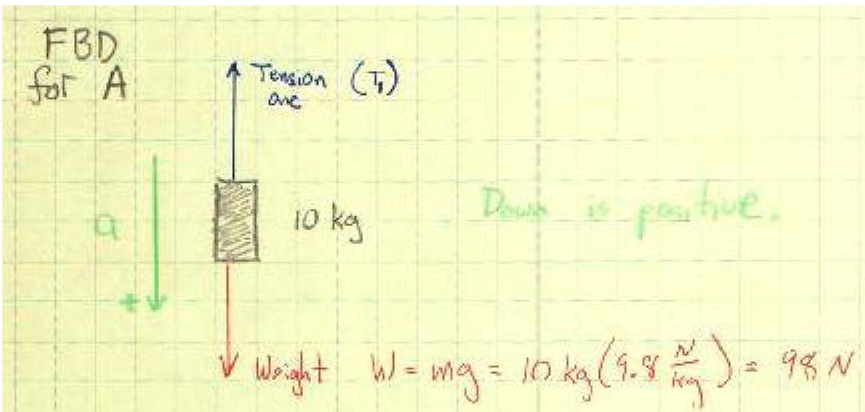
The value for a makes sense – it has to be less than 9.8 m/s^2 (that would be free-fall) and more than zero. And the units worked out, too.

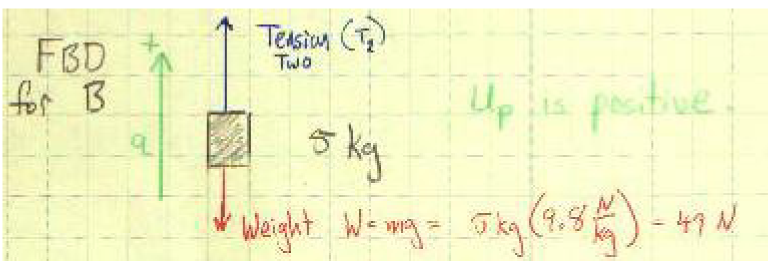
Example Problem #2:

Re-work the previous problem, except now the pulley has mass 2 kg and diameter 15 cm (it is still frictionless).

This problem asks for acceleration so we know we will have to use forces and Newton's laws to solve it – not Energy or Momentum. Nothing that we've done with Energy or Momentum has involved acceleration. So far, same as Example #1.

First we draw Free Body Diagrams for both masses:





Now things change from Example #1. The tensions will <u>not</u> be the same since the two blocks are essentially exerting forces the <u>pulley</u> via the string. The accelerations will be the same because the two blocks are connected by the string. One will accelerate upward and the other will accelerate downward.

Then we apply Newton's 2nd Law of motion to both. Note T_1 and T_2 this time.

Newton's 2nd Law For A:

$\Sigma F = ma$

$W - T_1 = ma$

$\boxed{98N - T_1 = (10\,kg)\,a}$

A

Newton's 2nd Law For B:

$\Sigma F = ma$

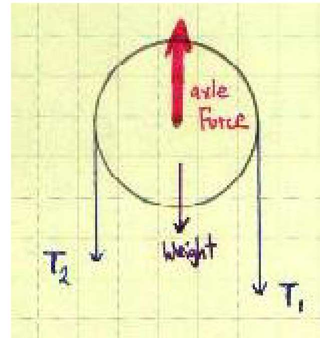
$T_2 - W = ma$

$\boxed{T_2 - 49N = (5\,kg)\,a}$

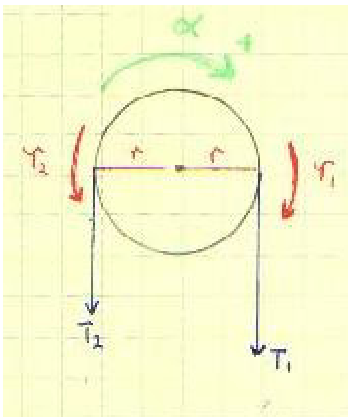
B

That gives two equations in <u>three</u> unknowns. We need more information.

Draw a FBD for the pulley.



However, the axle's force holding the pulley up is equal to the three downward forces – there is no net up or down motion of the pulley – it only rotates. So we'll look only at those forces that cause rotation – those forces that create torques:



Now I want to find the rotational inertia of the pulley:

Disk: $I = \frac{1}{2}MR^2$

$$I = \frac{1}{2}(2\,kg)(.075\,m)^2$$

$$I = .005625\,kgm^2$$

Now I'll apply Newton's 2nd Law for <u>rotation</u>:

$$\sum \tau = I\alpha \qquad a_t = r\alpha$$

$$\tau_1 - \tau_2 = I\alpha \qquad \alpha = \frac{a_t}{r} \qquad a_t = a$$

The acceleration of the edge of the pulley is the same as the acceleration of the blocks.

$$rT_1 - rT_2 = I\left(\frac{a}{r}\right)$$

$$(.075\,m)T_1 - (.075\,m)T_2 = \frac{(.005625\,kgm^2)a}{(.075\,m)}$$

$$T_1 - T_2 = \frac{(.005625\,kg\cancel{m^2})a}{(.075\,\cancel{m})(.075\,\cancel{m})}$$

$$T_1 - T_2 = (1\,kg)a$$

Now I have three equations in three unknowns. I can solve the problem:

Now there's 3 equations in 3 unknowns:

$$98N - T_1 = (10\,kg)a \longrightarrow 98N - (10kg)a = T_1$$

$$T_2 - 49N = (5\,kg)a \longrightarrow T_2 = (5kg)a + 49N$$

$$T_1 - T_2 = (1\,kg)a$$

Do the math:

Plug the first two into the last one:

$$\left(98N - (10kg)a\right) - \left((5kg)a + 49N\right) = (1\,kg)a$$

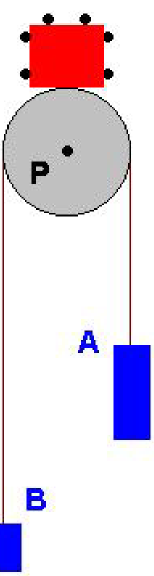
$$98N - (10kg)a - (5kg)a - 49N = (1\,kg)a$$

$$49\,N = 16\,kg\,(a)$$

$$\frac{49\,kg\,m/s^2}{16\,kg} = a$$

$$\boxed{3.06\,m/s^2 = a}$$

The value for a makes sense – it has to be less than 9.8 m/s^2 (that would be free-fall) and more than zero. And the units worked out, too. And it is less than the answer in the previous example. That makes sense because more mass is being accelerated, one way or another.



Example Problem #3:

An "Atwood's Machine" consists of two masses connected by a light string that passes over a pulley.

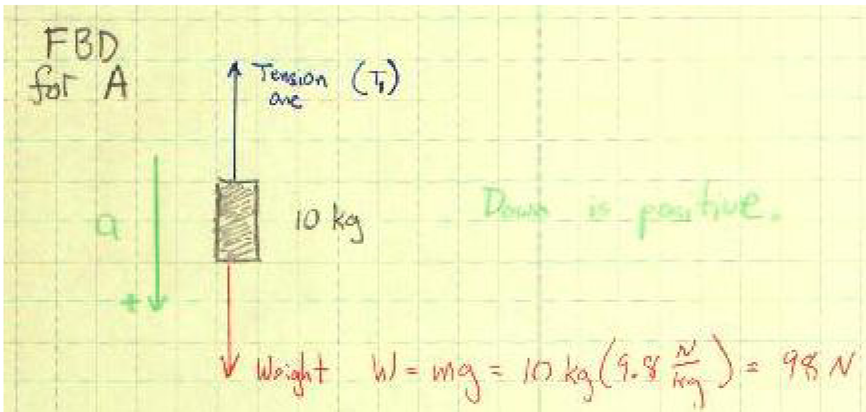
Mass A is 10 kg.
Mass B is 5 kg.

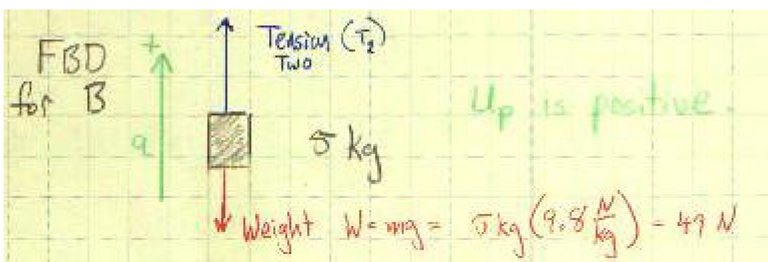
The pulley has mass 2 kg and diameter 15 cm.

A 5 kg block sits atop the pulley. It is held in place by pins so that it won't move side-to-side but its full weight rests on the pulley. The coefficient of friction between the block and the pulley is 0.5. Find the acceleration of the blocks.

This problem asks for acceleration so we know we will have to use forces and Newton's laws to solve it – not Energy or Momentum. Nothing that we've done with Energy or Momentum has involved acceleration.

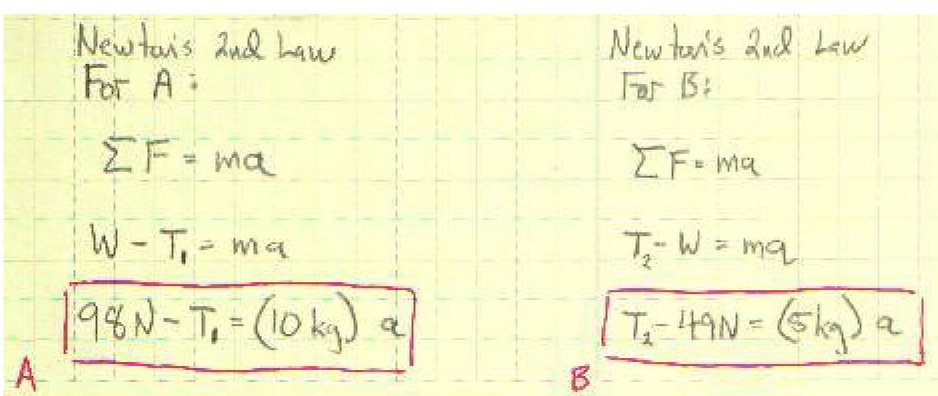
First we draw Free Body Diagrams for both masses:



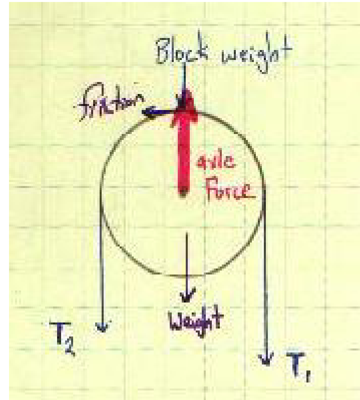


The tensions will not be the same since the two blocks are essentially exerting forces the pulley via the string. The accelerations will be the same because the two blocks are connected by the string. One will accelerate upward and the other will accelerate downward.

Then we apply Newton's 2nd Law of motion to both. Note T_1 and T_2 this time.

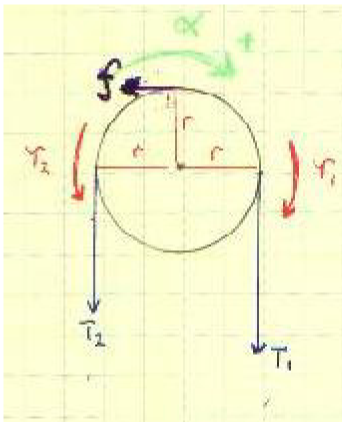


That gives two equations in <u>three</u> unknowns.



Draw a FBD for the pulley.

However, the axle's force holding the pulley up is equal to the four downward forces – there is no net up or down motion of the pulley – it only rotates. So we'll look only at those forces that cause rotation – those forces that create torques:

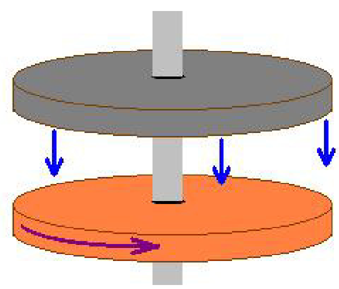


Note that this time I have a 3rd torque – one produced by the friction.

The rest of this is left as a homework problem.

Example Problem #4:

A disk of mass 500 kg, diameter .5 m is rotating on a turntable at 500 RPM.



A second disk, same diameter but 300 kg, drops on top the first disk. The second disk is not initially rotating. Friction acts between the two disks.

Find the final angular speed of the two disks in RPM. How much energy is turned to heat in this problem?

Lower Disk $m = 500\ kg$ $\quad \omega = 500 \frac{Rev}{Min}\left(\frac{2\pi}{1 Rev}\right)\left(\frac{1 min}{60 sec}\right) = 52.360 \frac{1}{sec}$

$r = .25\ m$

$I = \frac{1}{2} mr^2 = 15.625\ kgm^2$

Upper Disk $m = 300\ kg$ $\quad I = 9.375 \quad \omega = 0\ \frac{1}{sec}$

$r = .25\ m$

Conservation of Angular Momentum

Eventually both disks turn together. Both disks have same ω in the end.

$$L_{initial} = L_{final}$$

$$L_{upper} + L_{lower} = L_{upper} + L_{lower}$$

$$\underset{upper}{I\omega} + \underset{lower}{I\omega} = \underset{upper}{I\omega} + \underset{lower}{I_\omega}$$

$$9.375\ kgm^2\ (0) + 15.625\ (kgm^2)(52.360 \tfrac{1}{s}) = \omega\left(\underset{kgm^2}{9.375} + \underset{kgm^2}{15.625}\right)$$

$$0 + 818.125\left(\frac{kgm^2}{s}\right) = (25\ kgm^2)\ \omega$$

$$\frac{818.125 \frac{1}{s}}{25} = \omega$$

$$32.725 \tfrac{1}{s} = \omega = \boxed{313\ RPM}$$

Now lets look at energy. There is rotational KE in the system both before and after the 2nd disk drops.

Before: Upper disk $KE = \frac{1}{2} I \omega^2 = 0$

Lower disk $KE = \frac{1}{2} I \omega^2$

$$= \frac{1}{2}(15.625 \text{ kgm}^2)(52.360 \text{ 1/s})^2$$

$$= 21418.5125 \text{ kgm}^2/\text{s}^2$$

Before: TOTAL $E = 21418.5125$ J

After the lower dis

After the upper disk drops and the two are rotating at same ω:

Upper disk $KE = \frac{1}{2}(9.375 \text{ kgm}^2)(32.725 \text{ 1/s})^2$

$$= 5019.9639 \text{ J}$$

Lower disk $KE = \frac{1}{2}(15.625 \text{ kgm}^2)(32.725 \text{ 1/s})^2$

$$= 8366.6064 \text{ J}$$

After: Total $E = 5019.9639 + 8366.6064 = 13386.5704$ J

Heat generated is the KE lost:

$$\boxed{Q = E_{before} - E_{after} = 8032 \text{ J}}$$

Example Problem #5:

Think about what happens in the above problem if the coefficient of friction between the two disks is $\mu = 0.1$ (lubricated disks).

Think about what happens in the above problem if the coefficient of friction between the two disks is $\mu = 0.6$ (no lubrication between disks).

Do either of the final answers (i.e. 313 RPM; 8032 J) change if the μ value changes? If not, then what changes with the μ value?

The solution to Example #1 never involved μ, so μ can't have any effect on the final answers. The final speed is 313 RPM and the heat generated is 8032 J regardless of μ.

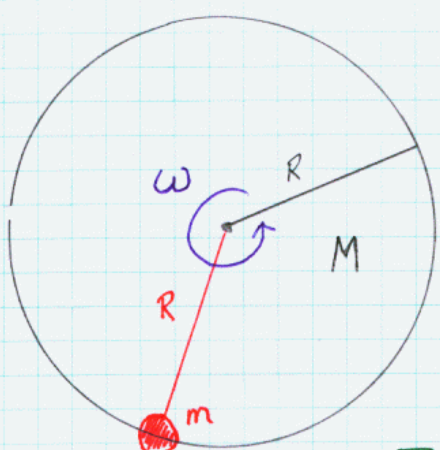
So what does μ change? Time. If μ is large then the two disks will match speeds sooner; if μ is small then the two disks match speeds after a longer time. The heat producing power is larger for larger μ because the same heat is produced in a shorter time period.

Example Problem #6:

A kid is riding on a merry-go-round. The merry-go-round is basically a rotating disk, and the kid is on its edge. The mass of the kid is half that of the merry-go-round. What happens to the angular velocity of the merry-go-round if the kid moves to it's center?

OK, first let me write down some things. The speed of the merry-go-round before the kid moves is going to be ω. The mass of the kid is m. The mass of the merry-go-round is M. $m = ½ M$. The new speed of the merry-go-round after the kid has move is ω_{new}.

Let's do it. I'm treating the merry-go-round as a disk of mass M and radius R. I have to look up the moment of inertia of a disk from the tables. The kid is an object of mass m sitting on the edge of the disk.



First I'll find the moment of inertias:

Moment of Inertia of Merry-Go-Round

$$I_{Disk} = \frac{1}{2}MR^2 \quad \text{(disk)}$$

Moment of Inertia of kid

$$I_{kid} = mR^2$$

Total I is

$$I_{TOTAL} = I_{Disk} + I_{kid}$$
$$= \frac{1}{2}MR^2 + mR^2$$
$$= \frac{1}{2}MR^2 + \frac{1}{2}MR^2$$

remember $m = \frac{1}{2}M$

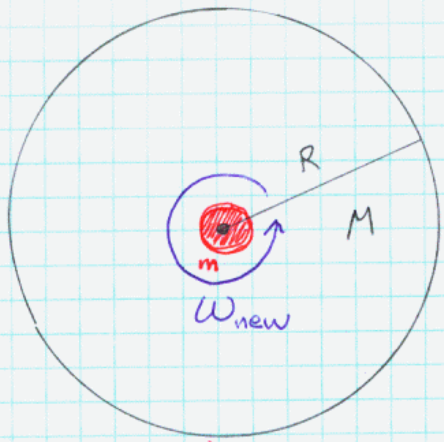
$$I_{total} = MR^2$$

I'll try using angular momentum to solve this.

The angular momentum is $L = I_{total}\,\omega$

$$L = MR^2\omega$$

Momentum is neither created or destroyed.



$I_{Disk} = \frac{1}{2} MR^2$

The disk is unchanged but the radius for the kid is now zero!

$I_{kid} = m(0)^2 = 0$

$$I_{total} = I_{disk} + I_{kid} = \frac{1}{2}MR^2 + 0$$

Angular momentum not created or destroyed

Previous angular momentum = New angular momentum

$$L = L_{new}$$

$$MR^2\omega = I_{total}\,\omega_{new}$$

$$MR^2\omega = \frac{1}{2}MR^2\,\omega_{new}$$

$$\omega = \frac{1}{2}\omega_{new}$$

$$\omega_{new} = 2\omega$$

So the merry-go-round is spinning twice as fast.

So if the merry-go-round was spinning at 30 RPM when the kid was on its edge it will be spinning at 60 RPM when the kid has moved to the center.

PHY 231 ONLY

Example Problem #7:

Find the moment of inertia of a rod of mass M and length L that is rotated about one end. What if the rod is rotated around its mid-point?

Put rod on a coordinate axis:

I know that $I = mr^2$ for an isolated small mass, so I'll break the rod up into isolated masses:

This little piece is located a distance x from the pivot, and it has mass dm and length dx.

The moment of inertia of this little mass is

$$dI = dm\, x^2$$

$$I = m\, r^2$$

To find total moment of inertia for the whole thing, add up all the little pieces:

$I = \int dI$ That is - integrate!

$I = \int dm\, x^2$ I must get dm in terms of x

$\frac{dm}{dx} = \frac{M}{L}$ ratio of mass to length

$$dm = \frac{M}{L}\,dx$$

So $I = \int \frac{M}{L} x^2\,dx$

$$I = \frac{M}{L}\int x^2\,dx$$

The rod extends from $x = 0$ to $x = L$, so those will be my limits of integration.

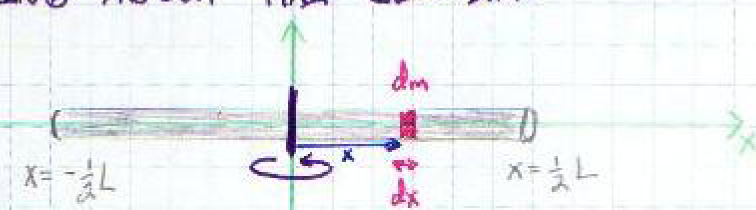
$$I = \frac{M}{L}\int_0^L x^2\,dx$$

$$I = \frac{M}{L}\left(\frac{x^3}{3}\right)\Big|_0^L$$

$$I = \frac{M}{L}\left(\frac{L^3}{3}\right) - \frac{M}{L}\left(\frac{0^3}{3}\right)$$

$$I = \frac{1}{3}ML^2$$

FOR ROTATING ABOUT THE CENTER:



Everything is the same, except the limits of integration will now be from $x = -\frac{1}{2}L$ to $x = \frac{1}{2}L$

$$I = \frac{M}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} x^2\,dx$$

$$I = \frac{M}{L}\left(\frac{x^3}{3}\right)\Bigg|_{-\frac{1}{2}L}^{\frac{1}{2}L}$$

$$I = \frac{M}{L}\left(\frac{\frac{1}{8}L^3}{3}\right) - \frac{M}{L}\left(\frac{-\frac{1}{8}L^3}{3}\right)$$

$$= \frac{1}{24}ML^2 + \frac{1}{24}ML^2 = \frac{2}{24}ML^2$$

$$I = \frac{1}{12}ML^2$$